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On the Boundary Value Problems of Hadamard Fractional Differential Equations of Variable Order via Kuratowski MNC Technique

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Abstract: In this manuscript, we examine both the existence and the stability of solutions of the boundary value problems of Hadamard-type fractional differential equations of variable order. New outcomes are obtained in this paper based on the Darbo's fixed point theorem (DFPT) combined with Kuratowski measure of noncompactness (KMNC). We construct an example to illustrate the validity of the observed results.

Keywords: derivatives and integrals of variable-order; boundary value problem; Darbo's fixed point theorem; measure of noncompactness; Ulam–Hyers–Rassias stability; Hadamard derivative

1. Introduction

The idea of fractional calculus is to replace the natural numbers in the derivative's order with rational ones. Although it seems an elementary consideration, it has an exciting correspondence explaining some physical phenomena.

Furthermore, studying both of the theoretical and practical aspects of fractional differential equations (FDEqs) has become a focus of an extensive international academic research [1–18]. A recent improvement in this investigation is the consideration of the notion of variable order operators. In this sense, various definitions of fractional operators involving the variable order have been introduced. This type of operators which are dependent on their power-law kernel can describe some hereditary specifications of numerous processes and phenomena [19,20]. In general, it is often difficult to find the analytical solution of FDEqs of variable order; therefore, numerical methods for the approximation of FDEqs of variable order are widespread. Regarding to the study existence of solutions to the problems of variable order, we refer to [21–26]. On the contrary, a consistent approach with the first-order precision for the solution of FDEqs of variable order is applied by Coimbra et al. in [27]. Lin et al. [28] discussed the convergence and stability of an explicit approximation related to the diffusion equation of variable order with a nonlinear source term. In [29], Zhuang et al. introduced the implicit and explicit Euler approximations for the nonlinear diffusion-advection equation of variable order.

While several research studies have been performed on investigating the solutions' existence of the fractional constant-order problems, the solutions' existence of the variable-order problems are rarely discussed in literature; we refer to [30–34]. Therefore, investigating this interesting special research topic makes all our results novel and worthy.

In particular, Agarwal et al. [5] studied the following problem:

$$\begin{cases} D_{0^+}^u x(t) = f(t, x(t)), \ t \in J := [0, \infty), \ u \in]1, 2], \\ x(0) = 0, \ x \text{ bounded on } [0, \infty), \end{cases}$$

where $D_{0^+}^u$ is the Riemann–Liouville fractional derivative of order u, f is a given function.

Inspired by [1–6,9,10,22–26,33,35], we deal with the following boundary value problem (BVP)

$$\begin{cases} {}^H D_{1^+}^{u(t)} x(t) = f_1(t, x(t)), \ t \in J := [1, T], \\ x(1) = x(T) = 0, \end{cases} \tag{1}$$

where $1 < u(t) \leq 2$, $f_1 : J \times X \to X$ is a continuous function and ${}^H D_{1^+}^{u(t)}, {}^H I_{1^+}^{u(t)}$ are the Hadamard fractional derivative and integral of variable-order $u(t)$.

The formal definitions and properties of the Hadamard fractional derivatives and integrals of variable-order will be given in the next section.

The goal of our research is to propose new existence criteria for the solutions of (1). In addition, we study the stability of the obtained solution of (1) in the sense of Ulam–Hyers–Rassias (UHR).

The remaining part of the paper is organized as follows. In Section 2, some notions and preliminaries are introduced. In Section 3, novel existence conditions are obtained based on the on the DFPT combined with KMNC. The UHR stability behavior is investigated in Section 4. In Section 5, to show the effectiveness of the obtained results, an example is considered. Section 6 is our Conclusions section.

2. Preliminaries

This section introduces some important fundamental definitions and concepts that will be needed for obtaining our results in the next sections.

The symbol $C(J, X)$ represents the Banach space of continuous functions $\kappa : J \to X$ with the norm

$$\|\kappa\| = Sup\{\|\kappa(t)\| : t \in J\},$$

where X is a real (or complex) Banach space.

2.1. Hadamard Fractional Integrals and Derivatives of Variable-Order: Definitions and Main Properties

For $0 < a_1 < a_2 < +\infty$, we consider the mappings $u(t) : [a_1, a_2] \to (0, +\infty)$ and $v(t) : [a_1, a_2] \to (n-1, n)$. Then, the left Hadamard fractional integral (HFI) of variable-order $u(t)$ for function $h_1(t)$ ([36,37]) is

$$ {}^H I_{a_1^+}^{u(t)} h_1(t) = \frac{1}{\Gamma(u(t))} \int_{a_1}^t (\log \frac{t}{s})^{u(t)-1} \frac{h_1(s)}{s} ds, \ t > a_1 \tag{2}$$

and the left Hadamard fractional derivative (HFD) of variable-order $v(t)$ for function $h_1(t)$ ([36,37]) is

$$ ({}^H D_{a_1^+}^{v(t)} h_1)(t) = \frac{1}{\Gamma(n-v(t))} (t\frac{d}{dt})^n \int_{a_1}^t (\log \frac{t}{s})^{n-v(t)-1} \frac{h_1(s)}{s} ds, \ t > a_1. \tag{3}$$

As anticipated, in case $u(t)$ and $v(t)$ are constant, then HFI and HFD coincide with the standard Hadamard integral and Hadamard derivative, respectively, see, e.g., [11,36,37].

Recall the following pivotal observation.

Lemma 1 ([11]). *Let α_1, $\alpha_2 > 0$, $a_1 > 1$, $h_1 \in L(a_1, a_2)$, and ${}^H D_{a_1^+}^{\alpha_1} h_1 \in L(a_1, a_2)$. Then, the differential equation*

$$ {}^H D_{a_1^+}^{\alpha_1} h_1 = 0 $$

has a unique solution

$$h_1(t) = \omega_1 (\log \frac{t}{a_1})^{\alpha_1-1} + \omega_2 (\log \frac{t}{a_1})^{\alpha_1-2} + \ldots + \omega_n (\log \frac{t}{a_1})^{\alpha_1-n},$$

and

$${}^H I_{a_1^+}^{\alpha_1} ({}^H D_{a_1^+}^{\alpha_1}) h_1(t) = h_1(t) + \omega_1 (\log \frac{t}{a_1})^{\alpha_1-1} + \omega_2 (\log \frac{t}{a_1})^{\alpha_1-2} + \ldots + \omega_n (\log \frac{t}{a_1})^{\alpha_1-n}$$

with $n-1 < \alpha_1 \leq n$, $\omega_\ell \in \mathbb{R}$, $\ell = 1, 2, \ldots, n$.
  Furthermore,

$${}^H D_{a_1^+}^{\alpha_1} ({}^H I_{a_1^+}^{\alpha_1}) h_1(t) = h_1(t)$$

and

$${}^H I_{a_1^+}^{\alpha_1} ({}^H I_{a_1^+}^{\alpha_2}) h_1(t) = {}^H I_{a_1^+}^{\alpha_2} ({}^H I_{a_1^+}^{\alpha_1}) h_1(t) = {}^H I_{a_1^+}^{\alpha_1+\alpha_2} h_1(t).$$

Remark 1. *Note that the semigroup property discussed in Lemma 1 is not fulfilled for general functions $u(t)$, $v(t)$, i.e., in general*

$${}^H I_{1^+}^{u(t)} ({}^H I_{1^+}^{v(t)}) h_1(t) \neq {}^H I_{1^+}^{u(t)+v(t)} h_1(t).$$

Example 1. *Let*

$$v(t) = \begin{cases} 1, & t \in [1,2] \\ 2, & t \in]2,4], \end{cases} \quad u(t) = \begin{cases} 2, & t \in [1,2] \\ 1, & t \in]2,4], \end{cases} \quad f_2(t) = 1, \ t \in [1,4],$$

$$\begin{aligned}
{}^H I_{1^+}^{u(t)} ({}^H I_{1^+}^{v(t)}) f_2(t) \ = \ & \frac{1}{\Gamma(2)} \int_1^2 \frac{1}{s} (\log \frac{t}{s})^1 \Big[\frac{1}{\Gamma(1)} \int_1^2 \frac{1}{\tau} (\log \frac{s}{\tau})^{1-1} d\tau + \frac{1}{\Gamma(2)} \int_2^s \frac{1}{\tau} (\log \frac{s}{\tau})^{2-1} d\tau \Big] ds \\
& + \frac{1}{\Gamma(1)} \int_2^t \frac{1}{s} (\log \frac{t}{s})^{1-1} \Big[\frac{1}{\Gamma(1)} \int_1^2 \frac{1}{\tau} (\log \frac{s}{\tau})^{1-1} d\tau + \frac{1}{\Gamma(2)} \int_2^s \frac{1}{\tau} (\log \frac{s}{\tau})^{2-1} d\tau \Big] ds
\end{aligned}$$

and

$${}^H I_{1^+}^{u(t)+v(t)} f_2(t) \ = \ \frac{1}{\Gamma(u(t)+v(t))} \int_1^t \frac{1}{s} (\log \frac{t}{s})^{u(t)+v(t)-1} f_2(s) ds.$$

Thus, we get

$$\begin{aligned}
{}^H I_{1^+}^{u(t)} ({}^H I_{1^+}^{v(t)}) f_2(t)|_{t=3} \ = \ & \frac{1}{\Gamma(2)} \int_1^2 \frac{1}{s} (\log \frac{3}{s}) \Big[\log 2 + \frac{1}{2} (\log \frac{s}{2})^2) \Big] ds + \frac{1}{\Gamma(1)} \int_2^3 \frac{1}{s} \Big[\log 2 + \frac{1}{2} (\log \frac{s}{2})^2 \Big] ds \\
\simeq \ & 0.9013
\end{aligned}$$

$$\begin{aligned}
{}^H I_{1^+}^{u(t)+v(t)} f_2(t)|_{t=3} \ = \ & \frac{1}{\Gamma(3)} \int_1^2 \frac{1}{s} (\log \frac{3}{s})^2 ds + \frac{1}{\Gamma(3)} \int_2^3 \frac{1}{s} (\log \frac{3}{s})^2 ds \\
\simeq \ & 0.2209
\end{aligned}$$

Therefore, we obtain

$${}^H I_{1^+}^{u(t)} ({}^H I_{1^+}^{v(t)}) f_2(t)|_{t=3} \neq {}^H I_{1^+}^{u(t)+v(t)} f_2(t)|_{t=3}.$$

Lemma 2. *If $u : J \to (1,2]$ is a continuous function, then, for $h_1 \in C_\delta(J,X) = \{h_1(t) \in C(J,X), (\log t)^\delta h_1(t) \in C(J,X)\}$, $(0 \leq \delta \leq 1)$, the variable order fractional integral ${}^H I_{1^+}^{u(t)} h_1(t)$ exists at any point in J.*

Proof. Taking the continuity of $\Gamma(u(t))$ into account, we shall claim that $M_u = \max_{t \in J} |\frac{1}{\Gamma(u(t))}|$ exists. We let $u^* = \max_{t \in J} |(u(t))|$. Thus, for $1 \le s \le t \le T$, we have

$$(\log \frac{t}{s})^{u(t)-1} \le 1, \quad \text{if} \quad 1 \le \frac{t}{s} \le e,$$

$$(\log \frac{t}{s})^{u(t)-1} \le (\log \frac{t}{s})^{u^*-1}, \text{ if } \quad \frac{t}{s} > e.$$

Then, for $1 \le \frac{t}{s} < +\infty$, we know

$$(\log \frac{t}{s})^{u(t)-1} \le \max\{1, (\log \frac{t}{s})^{u^*-1}\} = M^*.$$

For $h_1 \in C_\delta(J, X)$, by the definition of (2), we deduce that

$$
\begin{aligned}
|^H I_{1^+}^{u(t)} h_1(t)| &= \frac{1}{\Gamma(u(t))} \int_1^t (\log \frac{t}{s})^{u(t)-1} \frac{|h_1(s)|}{s} ds \\
&\le M_u \int_1^t (\log \frac{t}{s})^{u(t)-1} (\log s)^{-\delta} (\log s)^\delta \frac{|h_1(s)|}{s} ds \\
&\le M_u M^* \int_1^t \frac{1}{s} (\log s)^{-\delta} \max_{s \in J}(\log s)^\delta |h_1(s)| ds \\
&\le M_u M^* \max_{s \in J}(\log s)^\delta h^\star \int_1^t \frac{1}{s} (\log s)^{-\delta} ds \\
&\le M_u M^* \max_{s \in J}(\log s)^\delta h^\star \frac{(\log T)^{1-\delta}}{1-\delta} < \infty,
\end{aligned}
$$

where $h^\star = \max_{t \in J} |h_1(t)|$. It yields that the variable order fractional integral $^H I_{1^+}^{u(t)} h_1(t)$ exists at any point in J. $\square$

Lemma 3. *Let $u : J \to (1,2]$ be a continuous function. Then,*

$$^H I_{1^+}^{u(t)} h_1(t) \in C(J, X) \text{ for } h_1 \in C(J, X).$$

Proof. For $t, t_0 \in J$, $t_0 \le t$ and $h_1 \in C(J, X)$, we obtain

$$
\begin{aligned}
\left|^H I_{1^+}^{u(t)} h_1(t) -^H I_{1^+}^{u(t_0)} h_1(t_0) \right| &= \Big| \int_1^t \frac{1}{\Gamma(u(t))} (\log \frac{t}{s})^{u(t)-1} \frac{h_1(s)}{s} ds \\
&\quad - \int_1^{t_0} \frac{1}{\Gamma(u(t_0))} (\log \frac{t_0}{s})^{u(t_0)-1} \frac{h_1(s)}{s} ds \Big| \\
&= \Big| \int_0^1 \frac{1}{\Gamma(u(t))} \frac{(t-1)}{r(t-1)+1} (\log \frac{t}{r(t-1)+1})^{u(t)-1} h_1(r(t-1)+1) dr \\
&\quad - \int_0^1 \frac{1}{\Gamma(u(t_0))} \frac{(t_0-1)}{r(t_0-1)+1} (\log \frac{t_0}{r(t_0-1)+1})^{u(t_0)-1} h_1(r(t_0-1)+1) dr \Big| \\
&= \Big| \int_0^1 \Big[\frac{1}{\Gamma(u(t))} \frac{(t-1)}{r(t-1)+1} (\log \frac{t}{r(t-1)+1})^{u(t)-1} h_1(r(t-1)+1) \\
&\quad - \frac{1}{\Gamma(u(t))} \frac{(t_0-1)}{r(t_0-1)+1} (\log \frac{t}{r(t-1)+1})^{u(t)-1} h_1(r(t-1)+1) \Big] dr \\
&\quad + \int_0^1 \Big[\frac{1}{\Gamma(u(t))} \frac{(t_0-1)}{r(t_0-1)+1} (\log \frac{t}{r(t-1)+1})^{u(t)-1} h_1(r(t-1)+1) \\
&\quad - \frac{1}{\Gamma(u(t))} \frac{(t_0-1)}{r(t_0-1)+1} (\log \frac{t_0}{r(t_0-1)+1})^{u(t_0)-1} h_1(r(t-1)+1) \Big] dr
\end{aligned}
$$

$$
\begin{aligned}
+ & \int_0^1 \Big[\frac{1}{\Gamma(u(t))} \frac{(t_0-1)}{r(t_0-1)+1} (\log \frac{t_0}{r(t_0-1)+1})^{u(t_0)-1} h_1(r(t-1)+1) \\
- & \frac{1}{\Gamma(u(t_0))} \frac{(t_0-1)}{r(t_0-1)+1} (\log \frac{t_0}{r(t_0-1)+1})^{u(t_0)-1} h_1(r(t-1)+1) \Big] dr \\
+ & \int_0^1 \Big[\frac{1}{\Gamma(u(t_0))} \frac{(t_0-1)}{r(t_0-1)+1} (\log \frac{t_0}{r(t_0-1)+1})^{u(t_0)-1} h_1(r(t-1)+1) \\
- & \frac{1}{\Gamma(u(t_0))} \frac{(t_0-1)}{r(t_0-1)+1} (\log \frac{t_0}{r(t_0-1)+1})^{u(t_0)-1} h_1(r(t_0-1)+1) \Big] dr \Big|
\end{aligned}
$$

$$
\begin{aligned}
\leq \ & h^\star \int_0^1 \frac{1}{\Gamma(u(t))} (\log \frac{t}{r(t-1)+1})^{u(t)-1} \Big| \frac{(t-1)}{r(t-1)+1} - \frac{(t_0-1)}{r(t_0-1)+1} \Big| dr \\
+ \ & h^\star \int_0^1 \frac{1}{\Gamma(u(t))} \frac{(t_0-1)}{r(t_0-1)+1} \Big| (\log \frac{t}{r(t-1)+1})^{u(t)-1} - (\log \frac{t_0}{r(t_0-1)+1})^{u(t)-1} \Big| dr \\
+ \ & h^\star \int_0^1 \frac{(t_0-1)}{r(t_0-1)+1} (\log \frac{t_0}{r(t_0-1)+1})^{u(t_0)-1} \Big| \frac{1}{\Gamma(u(t))} - \frac{1}{\Gamma(u(t_0))} \Big| dr \\
+ \ & \int_0^1 \frac{1}{\Gamma(u(t_0))} \frac{(t_0-1)}{r(t_0-1)+1} (\log \frac{t_0}{r(t_0-1)+1})^{u(t_0)-1} \Big| h_1(r(t-1)+1) - h_1(r(t_0-1)+1) \Big| dr,
\end{aligned}
$$

where $h^\star = \max_{t \in J} |h_1(t)|$. On account of the continuity of the functions $\frac{(t-1)}{r(t-1)+1}$, $(\log \frac{t}{r(t-1)+1})^{u(t)-1}$, $\frac{1}{\Gamma(u(t))}$, $h_1(t)$, we get that the integral ${}^H I_{1^+}^{u(t)} h_1(t)$ is continuous at the point t_0. Since t_0 is arbitrary, we get that ${}^H I_{1^+}^{u(t)} h_1(t) \in C(J, X)$ for $h_1(t) \in C(J, X)$. $\square$

We will also use the following concepts from [34,38,39].

Definition 1. *I of $\mathbb{R}$ is called a generalized interval, if it is either an interval, or $\{a_1\}$ or $\{ \ \}$.*

Definition 2. *A finite set $\mathcal{P}$ is called a partition of I, if each x in I lies in exactly one of the generalized intervals $E \in \mathcal{P}$.*

Definition 3. *A function $g : I \to X$ is called piecewise constant with respect to partition $\mathcal{P}$ of I, if for any $E \in \mathcal{P}$, g is constant on E.*

2.2. Measures of Noncompactness

This subsection discusses some necessary background information about the KMNC.

Definition 4 ([40]). *Let X be a Banach space and Ω_X are bounded subsets of X. The KMNC is a mapping $\zeta : \Omega_X \to [0, \infty]$ which is constructed as follows:*

$$
\zeta(D) = \inf\{\epsilon > 0 : D(\in \Omega_X) \subseteq \cup_{\ell=1}^n D_\ell, \ diam(D_\ell) \leq \epsilon\},
$$

where

$$
diam(D_\ell) = \sup\{||x - y|| : \ x, y \in D_\ell\}.
$$

The following properties are valid for KMNC:

Proposition 1 ([40,41]). *Let X be a Banach space, D, D_1, D_2 are bounded subsets of X. Then,*
1. *$\zeta(D) = 0 \iff D$ is relatively compact.*
2. *$\zeta(\emptyset) = 0$.*
3. *$\zeta(D) = \zeta(\overline{D}) = \zeta(convD)$.*
4. *$D_1 \subset D_2 \implies \zeta(D_1) \leq \zeta(D_2)$.*
5. *$\zeta(D_1 + D_2) \leq \zeta(D_1) + \zeta(D_2)$.*

6. $\zeta(\lambda D) = |\lambda| \zeta(D), \lambda \in \mathbb{R}$.
7. $\zeta(D_1 \cup D_2) = Max\{\zeta(D_1), \zeta(D_2)\}$.
8. $\zeta(D_1 \cap D_2) = Min\{\zeta(D_1), \zeta(D_2)\}$.
9. $\zeta(D + x_0) = \zeta(D)$ for any $x_0 \in X$.

Lemma 4 ([42]). *If $U \subset C(J, X)$ is a equicontinuous and bounded set, then*

(i) the function $\zeta(U(t))$ is continuous for $t \in J$, and

$$\widehat{\zeta}(U) = \sup_{t \in J} \zeta(U(t)).$$

(ii) $\zeta\left(\int_0^T x(\theta)d\theta : x \in U\right) \leq \int_0^T \zeta(U(\theta))d\theta$, where

$$U(s) = \{x(s) : x \in U\}, \ s \in J.$$

Theorem 1 (DFPT [40]). *Let Λ be nonempty, closed, bounded and convex subset of a Banach space X and $W : \Lambda \longrightarrow \Lambda$ a continuous operator satisfying*

$$\zeta(W(S)) \leq k\zeta(S) \text{ for any } (S \neq \varnothing) \subset \Lambda, \ k \in [0, 1).$$

Then, W has at least one fixed point in Λ.

Definition 5 ([43]). *Let $\vartheta \in C(J, X)$. Equation (1) is UHR stable with respect to ϑ if there exists $c_f > 0$, such that for any $\epsilon > 0$ and for every solution $z \in C(J, X)$ of the following inequality:*

$$\|^H D_{1^+}^{u(t)} z(t) - f(t, z(t))\| \leq \epsilon \vartheta(t), \ t \in J. \tag{4}$$

there exists a solution $x \in C(J, X)$ of Equation (1) with

$$\|z(t) - x(t)\| \leq c_f \epsilon \vartheta(t), \ t \in J.$$

3. Main Existence Results

Let us introduce the following assumptions:

Hypothesis 1. *Let $n \in \mathbb{N}$ be an integer, $\mathcal{P} = \{J_1 := [1, T_1], J_2 := (T_1, T_2], J_3 := (T_2, T_3], \ldots J_n := (T_{n-1}, T]\}$ be a partition of the interval J, and let $u(t) : J \to (1, 2]$ be a piecewise constant function with respect to $\mathcal{P}$, i.e.,*

$$u(t) = \sum_{\ell=1}^{n} u_\ell I_\ell(t) = \begin{cases} u_1, & \text{if } t \in J_1, \\ u_2, & \text{if } t \in J_2, \\ \cdot \\ \cdot \\ \cdot \\ u_n, & \text{if } t \in J_n, \end{cases}$$

where $1 < u_\ell \leq 2$ are constants, and I_ℓ is the indicator of the interval $J_\ell := (T_{\ell-1}, T_\ell]$, $\ell = 1, 2, \ldots, n$, (here $T_0 = 1$, $T_n = T$) such that

$$I_\ell(t) = \begin{cases} 1, & \text{for } t \in J_\ell, \\ 0, & \text{elsewhere.} \end{cases}$$

Hypothesis 2. *Letting $(\log t)^\delta f_1 : J \times X \to X$ be a continuous function $(0 \leq \delta \leq 1)$, there exists a constant $K > 0$ such that*

$$(\log t)^\delta |f_1(t, x_1) - f_1(t, x_2)| \leq K|x_1 - x_2|$$

for any $x_1, x_2, \in X$ and $t \in J$.

Remark 2. *Using a remark in [44] page 20, we can easily show that the condition (H2) and the following inequality*

$$\zeta(\log t)^\delta \|f_1(t, B_1)\|) \leq K\zeta(B_1),$$

are equivalent for any bounded set $B_1 \subset X$ and for each $t \in J$.

Furthermore, for a given set U of functions $u : J \to X$, let us denote by

$$U(t) = \{u(t), u \in U\}, \ t \in J$$

and

$$U(J) = \{U(t) : v \in U, \ t \in J\}.$$

We are now in a position to prove the existence of solutions for the (BVP) (1) based on the concepts of MNCK and DFPT.

For each $\ell \in \{1, 2, \ldots, n\}$, the symbol $E_\ell = C(J_\ell, X)$, indicates the Banach space of continuous functions $x : J_\ell \to X$ equipped with the norm

$$\|x\|_{E_\ell} = \sup_{t \in J_\ell} \|x(t)\|,$$

where $\ell \in \{1, 2, \ldots, n\}$.

Using (3), the equation in the BVP (1) can be expressed as

$$\frac{1}{\Gamma(2 - u(t))} (t\frac{d}{dt})^2 \int_1^t (\log \frac{t}{s})^{1-u(t)} \frac{x(s)}{s} ds = f_1(t, x(t)), \quad t \in J. \tag{5}$$

Taking $(H1)$ into account, Equation (5) in the interval J_ℓ, $\ell = 1, 2, \ldots, n$ can be shown by

$$(t\frac{d}{dt})^2 \Big(\frac{1}{\Gamma(2 - u_1)} \int_1^{T_1} (\log \frac{t}{s})^{1-u_1} \frac{x(s)}{s} ds + \ldots$$

$$+ \frac{1}{\Gamma(2 - u_\ell)} \int_{T_{\ell-1}}^t (\log \frac{t}{s})^{1-u_\ell} \frac{x(s)}{s} ds \Big) = f_1(t, x(t)), \quad t \in J_\ell. \tag{6}$$

In what follows, we shall introduce the solution to the BVP (1).

Definition 6. *A function x_ℓ, $\ell = 1, 2, \ldots, n$, is a solution of the BVP (1) if $x_\ell \in C([1, T_\ell], X)$, x_ℓ satisfies (6) and $x_\ell(1) = 0 = x_\ell(T_\ell)$.*

According to the observation above, the BVP (1) can be expressed as in (5), with $J_\ell, \ell \in \{1, 2, \ldots, n\}$ as (6).

For $1 \leq t \leq T_{\ell-1}$, we take $x(t) \equiv 0$; then, (6) is written as

$${}^H D_{T_{\ell-1}^+}^{u_\ell} x(t) = f_1(t, x(t)), \quad t \in J_\ell.$$

We shall deal with the following BVP:

$$\begin{cases} {}^H D_{T_{\ell-1}^+}^{u_\ell} x(t) = f_1(t, x(t)), \quad t \in J_\ell \\ x(T_{\ell-1}) = 0, x(T_\ell) = 0. \end{cases} \tag{7}$$

For our purpose, the upcoming lemma will be a corner stone of the solution of (7).

Lemma 5. *A function $x \in E_\ell$ forms a solution of (7), if and only if x fulfills the integral equation*

$$x(t) = -(\log \frac{T_\ell}{T_{\ell-1}})^{1-u_\ell} (\log \frac{t}{T_{\ell-1}})^{u_\ell-1} {}^H I_{T_{\ell-1}^+}^{u_\ell} f_1(T_\ell, x(T_\ell)) + {}^H I_{T_{\ell-1}^+}^{u_\ell} f_1(t, x(t)). \tag{8}$$

Proof. We presume that $x \in E_\ell$ is solution of the BVP (7). Employing the operator ${}^H I^{u_\ell}_{T^+_{\ell-1}}$ to both sides of (7) and regarding Lemma 1, we find

$$x(t) = \omega_1 (\log \frac{t}{T_{\ell-1}})^{u_\ell - 1} + \omega_2 (\log \frac{t}{T_{\ell-1}})^{u_\ell - 2} + {}^H I^{u_\ell}_{T^+_{\ell-1}} f_1(t, x(t)), \quad t \in J_\ell.$$

Due to the assumption of function f_1 together with $x(T_{\ell-1}) = 0$, we conclude that $\omega_2 = 0$. Let $x(t)$ satisfy $x(T_\ell) = 0$. Thus, we observe that

$$\omega_1 = -(\log \frac{T_\ell}{T_{\ell-1}})^{1-u_\ell} {}^H I^{u_\ell}_{T^+_{\ell-1}} f_1(T_\ell, x(T_\ell)).$$

Then, we find

$$x(t) = -(\log \frac{T_\ell}{T_{\ell-1}})^{1-u_\ell} (\log \frac{t}{T_{\ell-1}})^{u_\ell - 1} {}^H I^{u_\ell}_{T^+_{\ell-1}} f_1(T_\ell, x(T_\ell)) + {}^H I^{u_\ell}_{T^+_{\ell-1}} f_1(t, x(t)), \quad t \in J_\ell.$$

Conversely, let $x \in E_\ell$ be a solution of integral Equation (8). Regarding the continuity of the function $(\log t)^\delta f_1$ and Lemma 1, we deduce that x is a solution of the BVP (7). □

Our novel existence result is presented in the next Theorem.

Theorem 2. *Assume that conditions (H1), (H2) hold and*

$$\frac{K[(\log T_\ell)^{1-\delta} - (\log T_{\ell-1})^{1-\delta}]}{(1-\delta)\Gamma(u_\ell)} (\log \frac{T_\ell}{T_{\ell-1}})^{u_\ell - 1} < \frac{1}{2}. \tag{9}$$

Then, the BVP (7) possesses at least one solution on J.

Proof. We construct the operator
$$W : E_\ell \to E_\ell,$$
as follows:

$$Wx(t) = -\frac{1}{\Gamma(u_\ell)} (\log \frac{T_\ell}{T_{\ell-1}})^{1-u_\ell} (\log \frac{t}{T_{\ell-1}})^{u_\ell - 1} \int_{T_{\ell-1}}^{T_\ell} (\log \frac{T_\ell}{s})^{u_\ell - 1} \frac{f_1(s, x(s))}{s} ds$$

$$+ \frac{1}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{t} (\log \frac{t}{s})^{u_\ell - 1} \frac{f_1(s, x(s))}{s} ds. \tag{10}$$

It follows from the properties of fractional integrals and from the continuity of function $(\log t)^\delta f_1$ that the operator $W : E_\ell \to E_\ell$ in (10) is well defined.
Let

$$R_\ell \geq \frac{\frac{2f^\star}{\Gamma(u_\ell + 1)} (\log \frac{T_\ell}{T_{\ell-1}})^{u_\ell}}{1 - \frac{2K[(\log T_\ell)^{1-\delta} - (\log T_{\ell-1})^{1-\delta}]}{(1-\delta)\Gamma(u_\ell)} (\log \frac{T_\ell}{T_{\ell-1}})^{u_\ell - 1}}$$

with

$$f^\star = \sup_{t \in J_\ell} |f_1(t, 0)|.$$

We consider the set
$$B_{R_\ell} = \{x \in E_\ell, \|x\|_{E_\ell} \leq R_\ell\}.$$

Clearly, B_{R_ℓ} is nonempty, closed, convex and bounded. Now, we demonstrate that W satisfies the assumption of the Theorem 1. We shall prove it in four phases:
Step 1: Claim: $W(B_{R_\ell}) \subseteq (B_{R_\ell})$.
For $x \in B_{R_\ell}$ and by (H2), we get

$$
\begin{aligned}
\|Wx(t)\| \;\le\;& \tfrac{1}{\Gamma(u_\ell)}(\log \tfrac{T_\ell}{T_{\ell-1}})^{1-u_\ell}(\log \tfrac{t}{T_{\ell-1}})^{u_\ell-1}\int_{T_{\ell-1}}^{T_\ell}(\log \tfrac{T_\ell}{s})^{u_\ell-1}\|f_1(s,x(s))\|\tfrac{ds}{s}\\
+\;& \tfrac{1}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{t}(\log \tfrac{t}{s})^{u_\ell-1}\|f_1(s,x(s))\|\tfrac{ds}{s}\\
\le\;& \tfrac{2}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{T_\ell}(\log \tfrac{T_\ell}{s})^{u_\ell-1}\|f_1(s,x(s))\|\tfrac{ds}{s}\\
\le\;& \tfrac{2}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{T_\ell}(\log \tfrac{T_\ell}{s})^{u_\ell-1}\|f_1(s,x(s))-f_1(s,0)\|\tfrac{ds}{s}+\tfrac{2}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{T_\ell}(\log \tfrac{T_\ell}{s})^{u_\ell-1}\|f_1(s,0)\|\tfrac{ds}{s}\\
\le\;& \tfrac{2}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{T_\ell}(\log \tfrac{T_\ell}{s})^{u_\ell-1}(\log s)^{-\delta}(K\|x(s)\|)\tfrac{ds}{s}+\tfrac{2f^\star}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{T_\ell}(\log \tfrac{T_\ell}{s})^{u_\ell-1}\tfrac{ds}{s}\\
\le\;& \tfrac{2K}{\Gamma(u_\ell)}(\log \tfrac{T_\ell}{T_{\ell-1}})^{u_\ell-1}\|x\|_{E_\ell}\int_{T_{\ell-1}}^{T_\ell}(\log s)^{-\delta}\tfrac{ds}{s}+\tfrac{2f^\star}{\Gamma(u_\ell+1)}(\log \tfrac{T_\ell}{T_{\ell-1}})^{u_\ell}\\
\le\;& \tfrac{2K[(\log T_\ell)^{1-\delta}-(\log T_{\ell-1})^{1-\delta}]}{(1-\delta)\Gamma(u_\ell)}(\log \tfrac{T_\ell}{T_{\ell-1}})^{u_\ell-1}R_\ell+\tfrac{2f^\star}{\Gamma(u_\ell+1)}(\log \tfrac{T_\ell}{T_{\ell-1}})^{u_\ell}\\
\le\;& R_\ell,
\end{aligned}
$$

which means that $W(B_{R_\ell})\subseteq (B_{R_\ell})$.

Step 2: Claim: W is continuous.

We presume that the sequence (x_n) converges to x in E_ℓ. Then,

$$
\begin{aligned}
&\|(Wx_n)(t)-(Wx)(t)\|\\
\le\;& \tfrac{1}{\Gamma(u_\ell)}(\log \tfrac{T_\ell}{T_{\ell-1}})^{1-u_\ell}(\log \tfrac{t}{T_{\ell-1}})^{u_\ell-1}\int_{T_{\ell-1}}^{T_\ell}(\log \tfrac{T_\ell}{s})^{u_\ell-1}\|f_1(s,x_n(s),{}^H I_{T_{\ell-1}^+}^{u_\ell}x_n(s))-f_1(s,x(s))\|\tfrac{ds}{s}\\
+\;& \tfrac{1}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{t}(\log \tfrac{t}{s})^{u_\ell-1}\|f_1(s,x_n(s),{}^H I_{T_{\ell-1}^+}^{u_\ell}x_n(s))-f_1(s,x(s))\|\tfrac{ds}{s}\\
\le\;& \tfrac{2}{\Gamma(u_\ell)}(\log \tfrac{t}{T_{\ell-1}})^{u_\ell-1}\int_{T_{\ell-1}}^{T_\ell}\|f_1(s,x_n(s),{}^H I_{T_{\ell-1}^+}^{u_\ell}x_n(s))-f_1(s,x(s))\|\tfrac{ds}{s}\\
\le\;& \tfrac{2}{\Gamma(u_\ell)}(\log \tfrac{t}{T_{\ell-1}})^{u_\ell-1}\int_{T_{\ell-1}}^{T_\ell}(\log s)^{-\delta}(K\|x_n(s)-x(s)\|)\tfrac{ds}{s}\\
\le\;& \tfrac{2K}{\Gamma(u_\ell)}\|x_n-x\|_{E_\ell}(\log \tfrac{t}{T_{\ell-1}})^{u_\ell-1}\int_{T_{\ell-1}}^{T_\ell}(\log s)^{-\delta}\tfrac{ds}{s}\\
\le\;& \tfrac{2K[(\log T_\ell)^{1-\delta}-(\log T_{\ell-1})^{1-\delta}]}{(1-\delta)\Gamma(u_\ell)}(\log \tfrac{T_\ell}{T_{\ell-1}})^{u_\ell-1}\|x_n-x\|_{E_\ell},
\end{aligned}
$$

i.e., we obtain

$$\|(Wx_n)-(Wx)\|_{E_\ell}\to 0 \;\text{ as }\; n\to\infty.$$

Ergo, the operator W is continuous on E_ℓ.

Step 3: Claim: W is bounded and equicontinuous.

By Step 2, we have $W(B_{R_\ell})=\{W(x):x\in B_{R_\ell}\}\subset B_{R_\ell}$. Thus, for each $x\in B_{R_\ell}$, we have $\|W(x)\|_{E_\ell}\le R_\ell$. Hence, $W(B_{R_\ell})$ is bounded. It remains to indicate that $W(B_{R_\ell})$ is equicontinuous.

For $t_1,t_2\in J_\ell$, $t_1<t_2$ and $x\in B_{R_\ell}$, we have

$$
\begin{aligned}
&\|(Wx)(t_2)-(Wx)(t_1)\|\\
=\;& \Big\| -\tfrac{1}{\Gamma(u_\ell)}(\log \tfrac{T_\ell}{T_{\ell-1}})^{1-u_\ell}(\log \tfrac{t_2}{T_{\ell-1}})^{u_\ell-1}\int_{T_{\ell-1}}^{T_\ell}(\log \tfrac{T_\ell}{s})^{u_\ell-1}f_1(s,x(s))\tfrac{ds}{s}\\
+\;& \tfrac{1}{\Gamma(u_\ell)}(\log \tfrac{T_\ell}{T_{\ell-1}})^{1-u_\ell}(\log \tfrac{t_1}{T_{\ell-1}})^{u_\ell-1}\int_{T_{\ell-1}}^{T_\ell}(\log \tfrac{T_\ell}{s})^{u_\ell-1}f_1(s,x(s))\tfrac{ds}{s}\\
+\;& \tfrac{1}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{t_2}(\log \tfrac{t_2}{s})^{u_\ell-1}f_1(s,x(s))\tfrac{ds}{s}-\tfrac{1}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{t_1}(\log \tfrac{t_1}{s})^{u_\ell-1}f_1(s,x(s))\tfrac{ds}{s}\Big\|\\
\le\;& \tfrac{1}{\Gamma(u_\ell)}(\log \tfrac{T_\ell}{T_{\ell-1}})^{1-u_\ell}\Big((\log \tfrac{t_2}{T_{\ell-1}})^{u_\ell-1}-(\log \tfrac{t_1}{T_{\ell-1}})^{u_\ell-1}\Big)\int_{T_{\ell-1}}^{T_\ell}(\log \tfrac{T_\ell}{s})^{u_\ell-1}\|f_1(s,x(s))\|\tfrac{ds}{s}\\
+\;& \tfrac{1}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{t_1}\Big((\log \tfrac{t_2}{s})^{u_\ell-1}-(\log \tfrac{t_1}{s})^{u_\ell-1}\Big)\|f_1(s,x(s))\|\tfrac{ds}{s}+\tfrac{1}{\Gamma(u_\ell)}\int_{t_1}^{t_2}(\log \tfrac{t_2}{s})^{u_\ell-1}\|f_1(s,x(s))\|\tfrac{ds}{s}\\
\le\;& \tfrac{1}{\Gamma(u_\ell)}(\log \tfrac{T_\ell}{T_{\ell-1}})^{1-u_\ell}\Big((\log \tfrac{t_2}{T_{\ell-1}})^{u_\ell-1}-(\log \tfrac{t_1}{T_{\ell-1}})^{u_\ell-1}\Big)\int_{T_{\ell-1}}^{T_\ell}(\log \tfrac{T_\ell}{s})^{u_\ell-1}\|f_1(s,x(s))-f_1(s,0)\|\tfrac{ds}{s}
\end{aligned}
$$

$$+ \quad \frac{1}{\Gamma(u_\ell)}(\log \frac{T_\ell}{T_{\ell-1}})^{1-u_\ell}\left((\log \frac{t_2}{T_{\ell-1}})^{u_\ell-1} - (\log \frac{t_1}{T_{\ell-1}})^{u_\ell-1}\right)\int_{T_{\ell-1}}^{T_\ell}(\log \frac{T_\ell}{s})^{u_\ell-1}\|f_1(s,0)\|\frac{ds}{s}$$

$$+ \quad \frac{1}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{t_1}(\log \frac{t_2}{t_1})^{u_\ell-1}\|f_1(s,x(s)) - f_1(s,0)\|\frac{ds}{s} + \frac{1}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{t_1}(\log \frac{t_2}{t_1})^{u_\ell-1}\|f_1(s,0)\|\frac{ds}{s}$$

$$+ \quad \frac{1}{\Gamma(u_\ell)}\int_{t_1}^{t_2}(\log \frac{t_2}{s})^{u_\ell-1}\|f_1(s,x(s)) - f_1(s,0)\|\frac{ds}{s} + \frac{1}{\Gamma(u_\ell)}\int_{t_1}^{t_2}(\log \frac{t_2}{s})^{u_\ell-1}\|f_1(s,0)\|\frac{ds}{s}$$

$$\leq \quad \frac{1}{\Gamma(u_\ell)}(\log \frac{T_\ell}{T_{\ell-1}})^{1-u_\ell}\left((\log \frac{t_2}{T_{\ell-1}})^{u_\ell-1} - (\log \frac{t_1}{T_{\ell-1}})^{u_\ell-1}\right)\int_{T_{\ell-1}}^{T_\ell}(\log \frac{T_\ell}{s})^{u_\ell-1}(\log s)^{-\delta}(K\|x(s)\|)\frac{ds}{s}$$

$$+ \quad \frac{f^\star}{\Gamma(u_\ell)}(\log \frac{T_\ell}{T_{\ell-1}})^{1-u_\ell}\left((\log \frac{t_2}{T_{\ell-1}})^{u_\ell-1} - (\log \frac{t_1}{T_{\ell-1}})^{u_\ell-1}\right)\int_{T_{\ell-1}}^{T_\ell}(\log \frac{T_\ell}{s})^{u_\ell-1}\frac{ds}{s}$$

$$+ \quad \frac{1}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{t_1}(\log \frac{t_2}{t_1})^{u_\ell-1}(\log s)^{-\delta}(K\|x(s)\|)\frac{ds}{s} + \frac{f^\star}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{t_1}(\log \frac{t_2}{t_1})^{u_\ell-1}\frac{ds}{s}$$

$$+ \quad \frac{1}{\Gamma(u_\ell)}\int_{t_1}^{t_2}(\log \frac{t_2}{s})^{u_\ell-1}(\log s)^{-\delta}(K\|x(s)\|)\frac{ds}{s} + \frac{f^\star}{\Gamma(u_\ell)}\int_{t_1}^{t_2}(\log \frac{t_2}{s})^{u_\ell-1}\frac{ds}{s}$$

$$\leq \quad \frac{K}{\Gamma(u_\ell)}(\log \frac{T_\ell}{T_{\ell-1}})^{1-u_\ell}(\log \frac{T_\ell}{T_{\ell-1}})^{u_\ell-1}\left((\log \frac{t_2}{T_{\ell-1}})^{u_\ell-1} - (\log \frac{t_1}{T_{\ell-1}})^{u_\ell-1}\right)\|x\|_{E_\ell}\int_{T_{\ell-1}}^{T_\ell}(\log s)^{-\delta}\frac{ds}{s}$$

$$+ \quad \frac{f^\star}{\Gamma(u_\ell+1)}(\log \frac{T_\ell}{T_{\ell-1}})^{1-u_\ell}\left((\log \frac{t_2}{T_{\ell-1}})^{u_\ell-1} - (\log \frac{t_1}{T_{\ell-1}})^{u_\ell-1}\right)(\log \frac{T_\ell}{T_{\ell-1}})^{u_\ell}$$

$$+ \quad \frac{K((\log t_2)^{1-\delta} - (\log t_1)^{1-\delta})}{(1-\delta)\Gamma(u_\ell)}(\log \frac{t_2}{t_1})^{u_\ell-1}\|x\|_{E_\ell} + \frac{f^\star}{\Gamma(u_\ell)}(\log \frac{t_2}{t_1})^{u_\ell-1}(\log t_1 - \log T_{\ell-1})$$

$$+ \quad \frac{K((\log t_2)^{1-\delta} - (\log t_1)^{1-\delta})}{(1-\delta)\Gamma(u_\ell)}(\log \frac{t_2}{t_1})^{u_\ell-1}\|x\|_{E_\ell} + \frac{f^\star}{\Gamma(u_\ell+1)}(\log \frac{t_2}{t_1})^{u_\ell}$$

$$\leq \quad \frac{K((\log T_\ell)^{1-\delta} - (\log T_{\ell-1})^{1-\delta})}{(1-\delta)\Gamma(u_\ell)}\|x\|_{E_\ell}\left((\log \frac{t_2}{T_{\ell-1}})^{u_\ell-1} - (\log \frac{t_1}{T_{\ell-1}})^{u_\ell-1}\right)$$

$$+ \quad \frac{f^\star}{\Gamma(u_\ell+1)}(\log \frac{T_\ell}{T_{\ell-1}})\left((\log \frac{t_2}{T_{\ell-1}})^{u_\ell-1} - (\log \frac{t_1}{T_{\ell-1}})^{u_\ell-1}\right) + \frac{2K((\log t_2)^{1-\delta} - (\log t_1)^{1-\delta})}{(1-\delta)\Gamma(u_\ell)}(\log \frac{t_2}{t_1})^{u_\ell-1}\|x\|_{E_\ell}$$

$$+ \quad \frac{f^\star}{\Gamma(u_\ell)}(\log \frac{t_2}{t_1})^{u_\ell-1}(\log t_1 - \log T_{\ell-1}) + \frac{f^\star}{\Gamma(u_\ell+1)}(\log \frac{t_2}{t_1})^{u_\ell}$$

$$\leq \quad \left[\frac{K((\log T_\ell)^{1-\delta} - (\log T_{\ell-1})^{1-\delta})}{(1-\delta)\Gamma(u_\ell)}\|x\|_{E_\ell} + \frac{f^\star}{\Gamma(u_\ell+1)}(\log \frac{T_\ell}{T_{\ell-1}})\right]\left((\log \frac{t_2}{T_{\ell-1}})^{u_\ell-1} - (\log \frac{t_1}{T_{\ell-1}})^{u_\ell-1}\right)$$

$$+ \quad \left[\frac{2K((\log t_2)^{1-\delta} - (\log t_1)^{1-\delta})}{(1-\delta)\Gamma(u_\ell)}\|x\|_{E_\ell} + \frac{f^\star}{\Gamma(u_\ell)}(\log t_1 - log T_{\ell-1})\right](\log \frac{t_2}{t_1})^{u_\ell-1} + \frac{f^\star}{\Gamma(u_\ell+1)}(\log \frac{t_2}{t_1})^{u_\ell}$$

Therefore, $\|(Wx)(t_2) - (Wx)(t_1)\| \to 0$ as $|t_2 - t_1| \to 0$. It implies that $W(B_{R_\ell})$ is equicontinuous.

Step 4: Claim: W is a k-set contraction.

For $U \in B_{R_\ell}$, $t \in J_\ell$, we get

$$\begin{aligned}\zeta(W(U)(t)) &= \zeta((Wy)(t), y \in U) \\ &\leq \left\{\frac{1}{\Gamma(u_\ell)}(\log \frac{T_\ell}{T_{\ell-1}})^{1-u_\ell}(\log \frac{t}{T_{\ell-1}})^{u_\ell-1}\int_{T_{\ell-1}}^{T_\ell}(\log \frac{T_\ell}{s})^{u_\ell-1}\zeta f(s,x(s))ds \right. \\ &\left. \quad + \frac{1}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{t}(\log \frac{t}{s})^{u_\ell-1}\zeta f(s,x(s))ds, y \in U\right\}.\end{aligned}$$

Then, Remark 2 implies that, for each $s \in J_\ell$,

$$
\begin{aligned}
\zeta(W(U)(t)) \quad \leq \quad & \Big\{ \tfrac{1}{\Gamma(u_\ell)}(\log \tfrac{T_\ell}{T_{\ell-1}})^{1-u_\ell}(\log \tfrac{t}{T_{\ell-1}})^{u_\ell-1}\int_{T_{\ell-1}}^{T_\ell}(\log \tfrac{T_\ell}{s})^{u_\ell-1}[K\widehat{\zeta}(U)s^{-\delta}]ds \\
& +\tfrac{1}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{t}(\log \tfrac{t}{s})^{u_\ell-1}[K\widehat{\zeta}(U)s^{-\delta}]ds,\; y \in U \Big\}. \\
\leq \quad & \Big\{ \tfrac{1}{\Gamma(u_\ell)}(\log \tfrac{T_\ell}{T_{\ell-1}})^{u_\ell-1}\int_{T_{\ell-1}}^{T_\ell}[K\widehat{\zeta}(U)s^{-\delta}]ds \\
& +\tfrac{1}{\Gamma(u_\ell)}(\log \tfrac{T_\ell}{T_{\ell-1}})^{u_\ell-1}\int_{T_{\ell-1}}^{t}[K\widehat{\zeta}(U)s^{-\delta}]ds,\; y \in U \Big\}. \\
\leq \quad & \tfrac{K(\log T_\ell)^{1-\delta}-(\log T_{\ell-1})^{1-\delta}}{(1-\delta)\Gamma(u_\ell)}(\log \tfrac{T_\ell}{T_{\ell-1}})^{u_\ell-1}\widehat{\zeta}(U) + \tfrac{K(\log t)^{1-\delta}-(\log T_{\ell-1})^{1-\delta}}{(1-\delta)\Gamma(u_\ell)}(\log \tfrac{T_\ell}{T_{\ell-1}})^{u_\ell-1}\widehat{\zeta}(U) \\
\leq \quad & \tfrac{2K[(\log T_\ell)^{1-\delta}-(\log T_{\ell-1})^{1-\delta}]}{(1-\delta)\Gamma(u_\ell)}(\log \tfrac{T_\ell}{T_{\ell-1}})^{u_\ell-1}\widehat{\zeta}(U)
\end{aligned}
$$

Thus,

$$
\widehat{\zeta}(WU) \leq \frac{2K[(\log T_\ell)^{1-\delta}-(\log T_{\ell-1})^{1-\delta}]}{(1-\delta)\Gamma(u_\ell)}(\log \frac{T_\ell}{T_{\ell-1}})^{u_\ell-1}\widehat{\zeta}(U).
$$

From inequality (9), it follows that W is a k-set contraction. $\quad\square$

Remark 3. *Variable-order problems constitute a very important class of problems regarding their applications [20,37]. Existence results for such problems are reported in [30–34]. Theorem 1 offers a new existence result for two points BVPs of Hadamard-type fractional differential equations of variable order using DFPT together with KMNC. Our results complement the existing ones and contribute to the development of the fundamental theory of variable-order Hadamard fractional differential equations.*

4. UHR Stability

In this section, we will offer a UHR stability result for the BVP (1).

Theorem 3. *Assume that:*
(i) Assumptions (H1) and (H2) and (9) hold.
(ii) The function $\varphi \in C(J_\ell, \mathbb{R}_+)$ is increasing and there exists $\lambda_\varphi > 0$, such that, for each $t \in J_\ell$, we have

$$
{}^H I_{T_{\ell-1}+}^{u_\ell}\varphi(t) \leq \lambda_{\varphi(t)}\varphi(t).
$$

Then, the Equation (1) is UHR stable with respect to φ.

Proof. Let $z \in C(J_\ell, \mathbb{R})$ satisfies the following inequality:

$$
\|{}^H D_{T_{\ell-1}+}^{u_\ell}z(t) - f_1(t, z(t), {}^H I_{T_{\ell-1}^+}^{u_\ell}z(t))\| \leq \epsilon\varphi(t), t \in J_\ell. \tag{11}
$$

Let $y \in C(J_\ell, \mathbb{R})$ be a solution of the problem

$$
\begin{cases}
{}^H D_{T_{\ell-1}+}^{u_\ell}x(t) = f_1(t, x(t)), \; t \in J_\ell \\
x(T_{\ell-1}) = 0, \; x(T_\ell) = 0.
\end{cases}
$$

By using Lemma (5), we have

$$
\begin{aligned}
x(t) = \quad & - \quad \tfrac{1}{\Gamma(u_\ell)}(\log \tfrac{T_\ell}{T_{\ell-1}})^{1-u_\ell}(\log \tfrac{t}{T_{\ell-1}})^{u_\ell-1}\int_{T_{\ell-1}}^{T_\ell}(\log \tfrac{T_\ell}{s})^{u_\ell-1}\tfrac{f_1(s,x(s))}{s}ds \\
& + \quad \tfrac{1}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{t}(\log \tfrac{t}{s})^{u_\ell-1}\tfrac{f_1(s,x(s))}{s}ds.
\end{aligned}
$$

By integration of (11) and from (H3), we obtain

$$\begin{aligned}
\|z(t) &+ \frac{1}{\Gamma(u_\ell)}(\log \tfrac{T_\ell}{T_{\ell-1}})^{1-u_\ell}(\log \tfrac{t}{T_{\ell-1}})^{u_\ell-1}\int_{T_{\ell-1}}^{T_\ell}(\log \tfrac{T_\ell}{s})^{u_\ell-1}\frac{f_1(s,z(s),{}^H I_{T_{\ell-1}^+}^{u_\ell} z(s))}{s}ds\\
&- \frac{1}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^t (\log \tfrac{t}{s})^{u_\ell-1}\frac{f_1(s,z(s),{}^H I_{T_{\ell-1}^+}^{u_\ell} z(s))}{s}ds\|\\
&\le \epsilon\frac{1}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^t \frac{1}{s}(\log \tfrac{t}{s})^{u_\ell-1}\varphi(s)ds\\
&\le \epsilon\lambda_{\varphi(t)}\varphi(t).
\end{aligned}$$

On the other hand, for every $t \in J_\ell$, we have

$$\begin{aligned}
\|z(t)-x(t)\| &= \|z(t)+\frac{1}{\Gamma(u_\ell)}(\log \tfrac{T_\ell}{T_{\ell-1}})^{1-u_\ell}(\log \tfrac{t}{T_{\ell-1}})^{u_\ell-1}\int_{T_{\ell-1}}^{T_\ell}(\log \tfrac{T_\ell}{s})^{u_\ell-1}\frac{f_1(s,x(s))}{s}ds\\
&\quad - \frac{1}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^t (\log \tfrac{t}{s})^{u_\ell-1}\frac{f_1(s,x(s))}{s}ds\|\\
&\le \|z(t)+\frac{1}{\Gamma(u_\ell)}(\log \tfrac{T_\ell}{T_{\ell-1}})^{1-u_\ell}(\log \tfrac{t}{T_{\ell-1}})^{u_\ell-1}\int_{T_{\ell-1}}^{T_\ell}(\log \tfrac{T_\ell}{s})^{u_\ell-1}\frac{f_1(s,z(s),{}^H I_{T_{\ell-1}^+}^{u_\ell} z(s))}{s}ds\\
&\quad - \frac{1}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^t (\log \tfrac{t}{s})^{u_\ell-1}\frac{f_1(s,z(s),{}^H I_{T_{\ell-1}^+}^{u_\ell} z(s))}{s}ds\|\\
&\quad + \frac{1}{\Gamma(u_\ell)}(\log \tfrac{T_\ell}{T_{\ell-1}})^{1-u_\ell}(\log \tfrac{t}{T_{\ell-1}})^{u_\ell-1}\int_{T_{\ell-1}}^{T_\ell}(\log \tfrac{T_\ell}{s})^{u_\ell-1}\frac{\|f_1(s,z(s),{}^H I_{T_{\ell-1}^+}^{u_\ell} z(s))-f_1(s,x(s))\|}{s}ds\\
&\quad + \frac{1}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^t (\log \tfrac{t}{s})^{u_\ell-1}\frac{\|f_1(s,z(s),{}^H I_{T_{\ell-1}^+}^{u_\ell} z(s))-f_1(s,x(s))\|}{s}ds\\
&\le \lambda_{\varphi(t)}\epsilon\varphi(t) + \frac{1}{\Gamma(u_\ell)}(\log \tfrac{T_\ell}{T_{\ell-1}})^{1-u_\ell}(\log \tfrac{t}{T_{\ell-1}})^{u_\ell-1}\\
&\quad \int_{T_{\ell-1}}^{T_\ell}(\log \tfrac{T_\ell}{s})^{u_\ell-1}\frac{1}{s}(\log s)^{-\delta}(K\|z(s)-x(s)\|)ds\\
&\quad + \frac{1}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^t (\log \tfrac{t}{s})^{u_\ell-1}\frac{1}{s}(\log s)^{-\delta}(K\|z(s)-x(s)\|)ds\\
&\le \lambda_{\varphi(t)}\epsilon\varphi(t) + \frac{1}{\Gamma(u_\ell)}(\log \tfrac{t}{T_{\ell-1}})^{u_\ell-1}\int_{T_{\ell-1}}^{T_\ell}\frac{1}{s}(\log s)^{-\delta}(K\|z(s)-x(s)\|)ds\\
&\quad + \frac{1}{\Gamma(u_\ell)}(\log \tfrac{t}{T_{\ell-1}})^{u_\ell-1}\int_{T_{\ell-1}}^t \frac{1}{s}(\log s)^{-\delta}(K\|z(s)-x(s)\|)ds\\
&\le \lambda_{\varphi(t)}\epsilon\varphi(t) + \frac{K((\log T_\ell)^{1-\delta}-(\log T_{\ell-1})^{1-\delta})}{(1-\delta)\Gamma(u_\ell)}(\log \tfrac{t}{T_{\ell-1}})^{u_\ell-1}\|z-x\|_{E_\ell}\\
&\quad + \frac{K((\log t)^{1-\delta}-(\log T_{\ell-1})^{1-\delta})}{(1-\delta)\Gamma(u_\ell)}(\log \tfrac{t}{T_{\ell-1}})^{u_\ell-1}\|z-x\|_{E_\ell}\\
&\le \lambda_{\varphi(t)}\epsilon\varphi(t) + \frac{2K[(\log T_\ell)^{1-\delta}-(\log T_{\ell-1})^{1-\delta}]}{(1-\delta)\Gamma(u_\ell)}(\log \tfrac{t}{T_{\ell-1}})^{u_\ell-1}\|z-x\|_{E_\ell}.
\end{aligned}$$

Then,

$$\|z-y\|_{E_\ell}\left(1 - \frac{2K[(\log T_\ell)^{1-\delta}-(\log T_{\ell-1})^{1-\delta}]}{(1-\delta)\Gamma(u_\ell)}(\log \tfrac{T_\ell}{T_{\ell-1}})^{u_\ell-1}\right) \le \lambda_{\varphi(t)}\epsilon\varphi(t).$$

We obtain, for each $t \in J_\ell$, that

$$\begin{aligned}
\|z-y\|_{E_\ell} &\le \frac{\lambda_{\varphi(t)}\epsilon\varphi(t)}{1 - \frac{2K[(\log T_\ell)^{1-\delta}-(\log T_{\ell-1})^{1-\delta}]}{(1-\delta)\Gamma(u_\ell)}(\log \tfrac{T_\ell}{T_{\ell-1}})^{u_\ell-1}}\\
&= [1 - \frac{2K[(\log T_\ell)^{1-\delta}-(\log T_{\ell-1})^{1-\delta}]}{(1-\delta)\Gamma(u_\ell)}(\log \tfrac{T_\ell}{T_{\ell-1}})^{u_\ell-1}]^{-1}\lambda_{\varphi(t)}\epsilon\varphi(t)\\
&:= c_f\epsilon\varphi(t).
\end{aligned}$$

Hence, Equation (7) is UHR stable with respect to φ for each $\ell \in \{1,2,\ldots,n\}$, which implies that Equation (1) is UHR stable with respect to φ. $\square$

5. An Example

To demonstrate our results, we will present the following example. We deal with the following fractional boundary value problem:

$$\begin{cases} D_{1^+}^{u(t)}x(t) = \frac{7}{5\sqrt{\pi}}(\log t)^{u(t)} + \frac{(\log t)^{-\frac{1}{3}}}{t+3}x(t), & t \in J := [1,e],\\ x(1) = 0, \ x(e) = 0. \end{cases} \tag{12}$$

Let

$$f_1(t, x) = \frac{7}{5\sqrt{\pi}} (\log t)^{u(t)} + \frac{(\log t)^{-\frac{1}{3}}}{t+3} x(t), \ (t, x) \in [1, e] \times [0, +\infty).$$

$$u(t) = \begin{cases} 1.3, & t \in J_1 := [1, 2], \\ 1.7, & t \in J_2 :=]2, e]. \end{cases} \tag{13}$$

Then, we have

$$
\begin{aligned}
(\log t)^{\frac{1}{3}} |f_1(t, x_1) - f_1(t, x_2)| &= \left| \frac{1}{t+3} x_1(t) - \frac{1}{t+3} x_2(t) \right| \\
&\leq \frac{1}{t+3} |x_1(t) - x_2(t)| \\
&\leq \frac{1}{4} |x_1(t) - x_2(t)|.
\end{aligned}
$$

Ergo, (H2) holds with $\delta = \frac{1}{3}$, $K = \frac{1}{4}$.

By (13), the equation in the problem (12) is divided into two expressions as follows:

$$
\begin{cases}
D_{1^+}^{1.3} x(t) = \frac{7}{5\sqrt{\pi}} (\log t)^{1.3} + \frac{(\log t)^{-\frac{1}{3}}}{t+3} x(t), & t \in J_1, \\
D_{2^+}^{1.7} x(t) = \frac{7}{5\sqrt{\pi}} (\log t)^{1.7} + \frac{(\log t)^{-\frac{1}{3}}}{t+3} x(t), & t \in J_2.
\end{cases}
$$

For $t \in J_1$, the problem (12) is equivalent to the following problem:

$$
\begin{cases}
D_{1^+}^{1.3} x(t) = \frac{7}{5\sqrt{\pi}} (\log t)^{1.3} + \frac{(\log t)^{-\frac{1}{3}}}{t+3} x(t), & t \in J_1, \\
x(1) = 0, \ x(2) = 0.
\end{cases} \tag{14}
$$

Next, we will prove that the condition (9) is fulfilled. We have that

$$\frac{K[(\log T_1)^{1-\delta} - (\log T_0)^{1-\delta}]}{(1-\delta)\Gamma(u_1)} (\log \frac{T_1}{T_0})^{u_1-1} = \frac{1}{4} \frac{(\log 2)^{\frac{2}{3}}}{(\frac{2}{3})\Gamma(1.3)} (\log 2)^{0.3} \simeq 0.2935 < \frac{1}{2}.$$

Let $\varphi(t) = (\log t)^{\frac{1}{2}}$

$$
\begin{aligned}
{}^H I_{1^+}^{u_1} \varphi(t) &= \frac{1}{\Gamma(1.3)} \int_1^t (\log \frac{t}{s})^{1.3-1} \frac{(\log s)^{\frac{1}{2}}}{s} ds \\
&\leq \frac{1}{\Gamma(1.3)} \int_1^t (\log \frac{t}{s})^{0.3} \frac{1}{s} ds \\
&\leq \frac{0.75}{\Gamma(2.3)} (\log t)^{\frac{1}{2}} := \lambda_{\varphi(t)} \varphi(t).
\end{aligned}
$$

Thus, condition (H3) is satisfied for $\varphi(t) = (\log t)^{\frac{1}{2}}$ and $\lambda_{\varphi(t)} = \frac{0.75}{\Gamma(2.3)}$.

By Theorem 2, the problem (14) has a unique solution $x_1 \in E_1$, and, from Theorem 3, Equation (14) is UHR stable with respect to $\varphi(t)$.

For $t \in J_2$, the problem (12) can been written as follows:

$$
\begin{cases}
D_{2^+}^{1.7} x(t) = \frac{7}{5\sqrt{\pi}} (\log t)^{1.7} + \frac{(\log t)^{-\frac{1}{3}}}{t+3} x(t), & t \in J_2, \\
x(2) = 0, \ x(e) = 0.
\end{cases} \tag{15}
$$

We see that

$$\frac{K[(\log T_2)^{1-\delta} - (\log T_1)^{1-\delta}]}{(1-\delta)\Gamma(u_2)}(\log \frac{T_2}{T_1})^{u_2-1} = \frac{1}{4}\frac{1 - (\log 2)^{\frac{2}{3}}}{(\frac{2}{3})\Gamma(1.7)}(\log \frac{e}{2})^{0.7} \simeq 0.0391 < \frac{1}{2}.$$

Accordingly, condition (9) is achieved. Moreover,

$$
\begin{aligned}
^H I_{2^+}^{u_2}\varphi(t) &= \frac{1}{\Gamma(1.7)}\int_2^t (\log \frac{t}{s})^{1.7-1}\frac{(\log s)^{\frac{1}{2}}}{s}ds \\
&\leq \frac{1}{\Gamma(1.7)}\int_2^t (\log \frac{t}{s})^{0.7}\frac{1}{s}ds \\
&\leq \frac{1}{\Gamma(2.7)}(\log t)^{\frac{1}{2}} := \lambda_{\varphi(t)}\varphi(t).
\end{aligned}
$$

Thus, condition (H3) is fulfilled for $\varphi(t) = (\log t)^{\frac{1}{2}}$ and $\lambda_{\varphi(t)} = \frac{1}{\Gamma(2.7)}$.

On the account of Theorem 2, problem (15) possesses a solution $\tilde{x}_2 \in E_2$. Furthermore, Theorem 3 yields that (15) is UHR stable with respect to $\varphi(t)$.

It is known that

$$x_2(t) = \begin{cases} 0, & t \in J_1 \\ \tilde{x}_2(t), & t \in J_2. \end{cases}$$

As a result, by definition (6), the BVP (12) has a solution

$$x(t) = \begin{cases} x_1(t), & t \in J_1, \\ x_2(t) = \begin{cases} 0, & t \in J_1, \\ \tilde{x}_2(t), & t \in J_2. \end{cases} \end{cases}$$

In addition, by Theorem 3, Equation (12) is UHR stable.

6. Conclusions

With this paper, we contribute to the development of the existence theory of boundary value problems with variable-order Hadamard derivatives. The main existence results offered here are based on the Darbo's fixed point theorem and Kuratowski's measure of noncompactness. Ulam–Hyers–Rassias stability results are also established for the problem under consideration. Since equations with variable-order Hadamard derivative are of importance for the theory and applications, our results can be of interest to many researchers in the field. An example is also presented to illustrate the observed results.

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