

Article

Impulsive Fractional Differential Inclusions and Almost Periodic Waves

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Abstract: In the present paper, the concept of almost periodic waves is introduced to discontinuous impulsive fractional inclusions involving Caputo fractional derivative. New results on the existence and uniqueness are established by using the theory of operator semigroups, Hausdorff measure of noncompactness, fixed point theorems and fractional calculus techniques. Applications to a class of fractional-order impulsive gene regulatory network (GRN) models are proposed to illustrate the results.

Keywords: fractional differential inclusions; impulses; almost periodicity; fixed point; Hausdorff measure of noncompactness



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1. Introduction

Fractional-order differential systems are generalizations of ordinary-order differential systems and have been widely studied in recent years due to their large potential for applications in numerous domains of science, engineering and practice [1–4], including some very recent results [5]. In fact, fractional-order derivatives have some very important advantages over the integer-order ones, including non-local properties for most of the fractional-order derivatives, hereditary and memory effects. Due to their superiority, it is now well recognized that systems with fractional-order derivatives are more useful from the applications perspective and reflect the long-term dependence on real-world models.

Parallel to the development of the theory of fractional-order systems, fractional differential inclusions were also intensively investigated. Numerous fundamental and qualitative results related to applications of the fractional calculus approach to differential inclusions are available in the existing literature. See, for example, [6–10].

However, the concept of almost periodic solutions (waves) for fractional-order differential inclusions has just begun to be studied [11–13]. Since the investigations into the existence of non-periodic solutions and their properties are of great significance for fractional-order inclusions, the theory of almost periodic waves should be further developed. Indeed, it is very well known that pure periodic solutions do not exist for fractional-order problems [14].

In fact, due to the importance of the notion of almost periodicity, the qualitative theory for almost periodic functions and solutions related to integer-order systems is very well developed. See, for example, [15–19] and the bibliography therein. It is worth noting that the theory of almost periodicity is relatively well developed for fractional-order systems [20–22]. Indeed, the almost periodicity concept is more general than the periodicity notion and more natural for application. The numerous applications of this concept include combinations of periodic waves with different periods, taking seasonal effects on the model parameters, celestial mechanics systems with moving periods that are not commensurable and many more into account [15,19]. Due to their great relevance to reality and their numerous implementations, the almost periodicity is considered a very

important qualitative property of solutions. However, the relevant results for fractional inclusions are few [11–13]. The main goal of the present paper is to contribute to the development of this area.

On the other hand, the effects of some impulsive perturbations can greatly contribute to the qualitative properties of the solutions. Additionally, such abrupt changes in the behavior are common in numerous real-world problems. That is why the fundamental and qualitative theories, as well as the applications of discontinuous impulsive systems and inclusions, have been the subject of intensive investigation. In addition, the so-called impulsive control technique has been successfully adapted to many applied problems. For some excellent results in impulsive systems, impulsive inclusions and impulsive control strategies, we refer to [23–30].

Due to the great possibilities in terms of their applications, impulsive perturbations have been considered for fractional-order differential systems, and the theory of impulsive fractional differential systems has been also very well-studied. We will mention several publications that give basic results for such systems [31–37].

The almost periodic properties also have an important value in impulsive differential systems, and the analysis of the impulsive effects on the almost periodic behavior of solutions to such integer [38–42] and fractional-order [43–47] systems has received significant attention. Therefore, their effects on the almost periodic behavior of fractional-order inclusions should be further investigated.

Due to the superiority and generalizations achieved by fractional-order derivatives, impulsive fractional-order inclusions have also been investigated by some researchers [48–51]. However, no results on the almost periodic waves of fractional differential inclusions have been reported in the existing literature. The present paper will address questions regarding the existence and uniqueness of such solutions, since these are interesting and important open problems for theory and application.

The main contributions of this article are:

1. The important concept of almost periodic waves is introduced to the class of impulsive fractional-order inclusions;
2. The effects of considering impulsive perturbations and fractional-order derivatives on the almost periodic properties are investigated and criteria for existence and uniqueness are established;
3. The main results are obtained by applying the theory of operators semigroup, Hausdorff measure of noncompactness, fixed point theorems and techniques based on fractional calculus;
4. The efficiency of the obtained results is demonstrated on a fractional impulsive GRN model.

The remaining part of the paper is structured as follows. In Section 2, some notations from multi-valued analysis and fractional calculus, and some proprieties of almost sectorial operators are given. The research problem is formulated in this paper. The almost periodic concept is introduced to the impulsive fractional-order inclusion problem. In Section 3, sufficient conditions are obtained for the existence and uniqueness of almost periodic waves for fractional differential inclusions under impulsive perturbations. Section 4 is devoted to the application of the established results to a class of fractional impulsive GRN models. Finally, some concluding remarks are given in Section 5.

2. Preliminaries

Throughout this paper, we consider a real Banach space $(E; \|\cdot\|)$ with an arbitrary norm $\|\cdot\|$ and the Banach space $C_b(J, E)$ formed of all continuous and bounded functions $y(t) : J \rightarrow E$. Here, J denotes a subset of the set of all real numbers $\mathbb{R} = (-\infty, \infty)$ and allows \mathbb{R} to be endowed with the usual Lebesgue measure, defined on the Lebesgue σ -

algebra Λ . We also consider a class of Caputo fractional derivatives, where, according to [4], the Caputo fractional derivative of a function g of order $\alpha, 0 < \alpha < 1$ is defined as

$${}^c D^\alpha g(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t \frac{g'(s)}{(t-s)^\alpha} ds,$$

where g is an abstract continuous function on the interval $[a, b]$ and Γ stands for the Gamma function.

In this paper, we will consider a fractional order $0 < \alpha \leq 1$ semi-linear impulsive differential inclusion in the form

$$\begin{cases} {}^c D^\alpha y(t) \in Ay(t) + F(t, y(t)), \text{ a.e. } t \in \mathbb{R}, t \neq t_k, \\ \Delta y(t_k) = y(t_k^+) - y(t_k) = I_k(y(t_k)), k = \pm 1, \pm 2, \dots, \end{cases} \tag{1}$$

where $A : \mathcal{D}(A) \subset E$ is a generator of a C_0 -semigroup $\{S(t), t \geq 0\}$ in the Banach space $E, F(t, y)$ is a multi-function which contains the selection $S_{F(\cdot, y(\cdot))}$ of single-valued functions $f(t, y(t)), I_k(\cdot)$ are the multi-valued impulsive functions for $k = \pm 1, \pm 2, \dots$ and the impulsive instants $t_k, k = \pm 1, \pm 2, \dots$, are such that $t_k \in \mathcal{T}$, where the set \mathcal{T} of all unbounded and strictly increasing sequences of the type $\{t_k\}$ is defined as

$$\mathcal{T} = \{ \{t_k\} : t_k \in (-\infty, \infty), t_k < t_{k+1}, t_k \neq 0, k = \pm 1, \pm 2, \dots, \lim_{k \rightarrow \pm \infty} t_k = \pm \infty \},$$

$y(t_k^+) = \lim_{t \rightarrow t_k^+} y(t)$ is the right bounds of $y(t)$ at the points t_k , for every $k = \pm 1, \pm 2, \dots$

A function $y(t)$ is a solution of (1) [23], if a function $v(t)$ exists from the selector of $F(t, y(t))$ a.e. in \mathbb{R} , such that

$$\begin{cases} {}^c D^\alpha y(t) = Ay(t) + v(t), t \neq t_k, \\ y(t_k^+) = y(t_k) + a_k, \end{cases} \tag{2}$$

where $a_k = I_k(y(t_k)), k = \pm 1, \pm 2, \dots$

Remark 1. For more details of impulsive integer-order and fractional-order inclusions in Banach spaces, we refer to [23,50–52].

In the following, we will use the following norm $\|y(t)\|_\infty = \sup\{\|y(t)\|, t \in J\}$ and by $B(E)$, we denote the Banach space of all linear bounded operators M from E into E with the norm $\|M\|_{B(E)} = \sup\{\|M(y)\| : \|y\| = 1\}$.

A function $y : J \rightarrow E$ is summable, if, for every open subset $U \subset E$, the set $y^{-1}(U) = \{t \in J : y(t) \in U\}$ is Lebesgue-summable. It is well-known that a measurable function $y : J \rightarrow E$ is Bochner-summable, if $\|y\|$ is Lebesgue-summable [53,54].

In the paper, $L^1(J, E)$ denotes the Banach space of all functions $y(t) : J \rightarrow E$, which are Bochner integrable with the norm

$$\|y\|_1 = \int_J \|y(t)\| dt$$

and $L^1_+(J, \mathbb{R})$ stands for the space of all Lebesgue summable non-negative functions defined in J .

We will use the following notations [10,23]:

$$\begin{aligned} \mathcal{P}(E) &= \{U \subset E : U \neq \emptyset\}, \\ \mathcal{P}_{cl}(E) &= \{U \subset \mathcal{P}(E) : U \text{ closed}\}, \\ \mathcal{P}_b(E) &= \{U \in \mathcal{P}(E) : U \text{ bounded}\}, \\ \mathcal{P}_{cv}(E) &= \{U \in \mathcal{P}(E) : U \text{ convex}\}, \\ \mathcal{P}_{cp}(E) &= \{U \in \mathcal{P}(E) : U \text{ compact}\}. \end{aligned}$$

For any two metric spaces $(U; d_U)$ and $(H; d_H)$ with metrics d_U, d_H , respectively, and for the multi-valued map $G : U \rightarrow \mathcal{P}cl(H)$, we will use the notation S_G^1 for the set of summable selections of G . We recall that a map $G : U \rightarrow \mathcal{P}cl(H)$ is upper semicontinuous (u.s.c.) on U , if the set $G^{+1}(V) = \{x \in U, G(x) \in V\}$ is open for any open set $V \in H$.

We denote by $J_k = (t_k, t_{k+1}]$ and let $PC[\mathbb{R}, E] = \{y : \mathbb{R} \rightarrow E, y|_{J_k} \in C[J_k, E] \text{ and } y(t_k^+) \in E, k = \pm 1, \pm 2, \dots\}$ exists, where $C[J_k, E]$ is the set of all continuous functions defined on J_k which take values on E .

Additionally, let χ_k denote the Hausdorff measure of noncompactness on the Banach space $C[\bar{J}_k; E]$, where $\bar{J}_k = \overline{\text{co}}(J_k)$ means the closure of the convex hull of J_k and we will use the map $\chi_{PC} : \mathcal{P}_b(PC[\mathbb{R}, E]) \rightarrow [0, \infty)$, where

$$\chi_{PC}(B) = \sup_{k=\pm 1, \pm 2, \dots} \chi_k(B|_{\bar{J}_k}).$$

Obviously, the measure $\chi_{PC}(B)$ is the Hausdorff measure of noncompactness for the set $PC[\mathbb{R}, E]$ [52].

In the paper, we will need the next property.

Lemma 1 ([52,55]). *Let B be a bounded set in a real Banach space E . Then, for every $\varepsilon > 0$, there exists a sequence $\{x_n\}_{n \in \mathbb{N}} \in B$, such that*

$$\chi_{PC}(B) \leq 2\chi_{PC}(\{x_n\}, n \in \mathbb{N}) + \varepsilon.$$

We also will use the property -generalized Cantor’s intersection, i.e., if $\{B_n\}_{n=1}^\infty$ is a decreasing sequence of the nonempty, closed and bounded subset of $PC[\mathbb{R}, E]$ and $\lim_{n \rightarrow \infty} \chi(B_n)$, then $\bigcap_{n=1}^\infty B_n$ is not empty and compact.

Since the solutions of (1) belong to the space $PC[\mathbb{R}, E]$, we will now provide some definitions to introduce the concept of almost periodicity in the sense of Weyl [56].

Let $\varphi \in PC[\mathbb{R}, E]$, and for any $p \geq 1$ and $l > 0$

$$\|\varphi\|_{p,l}^{PC} = \left(\sup_{a \in \mathbb{R}} \frac{1}{l} \int_a^{a+l} \|\varphi(t)\|^p dt \right)^{\frac{1}{p}} < +\infty, \quad \|\varphi\|_p^{PC} = \lim_{l \rightarrow \infty} \|\varphi\|_{p,l}^{PC}$$

be the norm and seminorm in E , respectively. The number $\tau \in \mathbb{R}$ is called an ε – almost periods of the function φ , if $\|\varphi(t + \tau) - \varphi(t)\|_p^{PC} < \varepsilon$.

Definition 1. *A. The set of sequences $\{t_k^j\}, t_k^j = t_{k+j} - t_k, k, j = \pm 1, \pm 2, \dots$ is said to be uniformly almost periodic if, for an arbitrary $\varepsilon > 0$, there exists a relatively dense set T of ε – almost periods common for any sequences.*

B. The function $\varphi \in PC[\mathbb{R}, E]$ is said to be almost periodic in the sense of Weyl piecewise function, if:

- B1 The set of sequences $\{t_k^j\}$ is uniformly almost periodic;
- B2 For any $\varepsilon > 0$ there exists a real number $\delta > 0$ such that if the points t' and t'' belong to the same interval of continuity of, $\varphi(t)$, and satisfy the inequality $|t' - t''| < \delta$, then $\|\varphi(t') - \varphi(t'')\|_p^{PC} < \varepsilon$;
- B3 For any $\varepsilon > 0$, there exists a relatively dense set T , such that if $\tau \in T$, then $\|\varphi(t + \tau) - \varphi(t)\|_p^{PC} < \varepsilon$ for all $t \in \mathbb{R}$ satisfying the condition $|t - t_k| > \varepsilon, k = \pm 1, \pm 2, \dots$

Consider the next assumptions for almost periodicity.

- A1. The set of sequences $\{t_k^j\}, t_k^j = t_{k+j} - t_k, k, j = \pm 1, \pm 2, \dots$ is uniformly almost periodic, and there exists $\theta > 0$ such that $\inf_k t_k^1 = \theta > 0$;
- A2. The function $v \in PC[\mathbb{R}, E]$ is almost periodic in the sense of Weyl piecewise function;
- A3. The sequence $\{a_k\}, a_k \in E, k = \pm 1, \pm 2, \dots$ is almost periodic.

It follows from assumption A1 [42] that the points $\{t_k\}$ are uniformly distributed along \mathbb{R} and, for each $l > 0$, there exists a positive integer N , such that each interval of length l has no more than N elements of the sequence $\{t_k\}$, i.e.,

$$i(s, t) \leq N(t - s) + N, \tag{3}$$

where $i(s, t)$ is the number of points t_k in the interval (s, t) .

Following similar results from [41,42] we can prove the next auxiliary results.

Lemma 2. *Let the assumptions A1–A3 hold.*

Then, for each $\varepsilon > 0$, there exist $\varepsilon_1, 0 < \varepsilon_1 < \varepsilon$ and relatively dense sets T of real numbers, and Q of integer numbers, such that the following relations are fulfilled:

- (a) $\|v(t + \tau) - v(t)\|_p^{PC} < \varepsilon, t \in \mathbb{R}, \tau \in T, |t - t_k| > \varepsilon, k = \pm 1, \pm 2, \dots;$
- (b) $\|a_{k+q} - a_k\| < \varepsilon, q \in Q, k = \pm 1, \pm 2, \dots;$
- (c) $|t_{k+q} - \tau| < \varepsilon_1, q \in Q, \tau \in T, k = \pm 1, \pm 2, \dots$

Let the operator $-A$ in (1) and (2) be an infinitesimal generator of an analytic semi-group $S(t)$ in the Banach space E and $0 \in \rho(A)$, $\rho(A)$ be the resolvent set of A . For any $\beta > 0$, we define the fractional power $A^{-\beta}$ of the operator A by

$$A^{-\beta} = \frac{1}{\Gamma(\beta)} \int_0^\infty t^{\beta-1} S(t) dt,$$

where $A^{-\beta}$ is bounded, bijective and $A^\beta = (A^{-\beta})^{-1}, \beta > 0$ is a closed linear operator such that $\mathcal{D}(A^\beta) = \mathcal{R}(A^{-\beta}), \mathcal{R}(A^{-\beta})$ is the range of $A^{-\beta}$. The operator A^0 is the identity operator in E and for $0 \leq \beta \leq 1$, the space $E_\beta = \mathcal{D}(A^\beta)$ with norm $\|y\|_\beta = \|A^\beta y\|$ is a Banach space [4,21,57–60].

The following results will be useful in the proof of our main results.

Lemma 3 ([53]). *Let $-A$ be an infinitesimal operator of an analytic semigroup $S(t)$. Then:*

- (a) $S(t) : E \rightarrow \mathcal{D}(A^\beta)$ for every $t > 0$, and $\beta \geq 0$;
- (b) For every $y \in \mathcal{D}(A^\beta)$, the following equality $S(t)A^\beta y = A^\beta S(t)y$ holds;
- (c) For every $t > 0$, the operator $A^\beta S(t)$ is bounded and

$$\|A^\beta S(t)\| \leq K_\beta t^{-\beta} e^{-\lambda t}, K_\beta > 0, \lambda > 0;$$

- (d) For $0 < \beta \leq 1$ and $y \in \mathcal{D}(A^\beta)$, we have

$$\|S(t)y - y\| \leq C_\beta t^\beta \|A^\beta y\|, C_\beta > 0.$$

In the future considerations, we will use the notations

$$\zeta = \min\{\theta(t - t_k)^\alpha, 0 < t - t_k \leq 1\}$$

and

$$L(A) = K_\beta \left[\frac{\pi}{\Gamma(\beta) \sin \pi\beta\lambda^{1-\beta}} + 2N \left(\frac{1}{\zeta^\beta} + \frac{1}{e^\beta - 1} \right) \right] > 0.$$

When $-A$ generates a semigroup with a negative exponent, we assume that if $\phi(t)$ is a bounded solution of (2) on \mathbb{R} , then

$$\phi(t) = \int_{-\infty}^t (t - s)^{\alpha-1} \mathcal{S}(t - s)v(s)ds + \sum_{t_k < t} \mathcal{T}(t - t_k)a_k, \tag{4}$$

where

$$\mathcal{T}(t) = \int_0^\infty \xi_\alpha(\theta)S(t^\alpha\theta)d\theta, \mathcal{S}(t) = \alpha \int_0^\infty \theta \xi_\alpha(\theta)S(t^\alpha\theta)d\theta,$$

$$\zeta_\alpha(\theta) = \frac{1}{\alpha} \theta^{-1-\frac{1}{\alpha}} \omega_\alpha(\theta^{-\frac{1}{\alpha}}) \geq 0,$$

$$\omega_\alpha(\theta) = \frac{1}{\pi} \sum_{n=1}^{\infty} (-1)^{n-1} \theta^{-n\alpha-1} \frac{\Gamma(n\alpha+1)}{n!} \sin(n\pi\alpha), \theta \in (0, \infty),$$

ζ_α is a probability density function defined on $(0, \infty)$, that is

$$\zeta_\alpha \geq 0, \theta \in (0, \infty) \text{ and } \int_0^\infty \zeta_\alpha(\theta) d\theta = 1.$$

The next lemma is similar to Proposition 2.2 in [52] for the case of operator $-A$ and semigroup $S(t)$. Similar results can be found in [61].

Lemma 4. Let χ be the Hausdorff measure of noncompactness on E and $\{f_m\}$, $m \in \mathbb{N}$ be a sequence in $L^1(\mathbb{R}, E)$, such that there exist $\eta, \zeta \in L^1_+(\mathbb{R}, E)$ with the properties:

- (i) $\sup_{m \in \mathbb{N}} f_m(t) \leq \eta(t)$ a.e. $t \in \mathbb{R}$;
- (ii) $\chi(\{f_m\}, m \in \mathbb{N}) \leq \zeta(t)$ a.e. $t \in \mathbb{R}$.

Then, for every $t \in \mathbb{R}$, we have

$$\chi\left(\left\{\int_0^t (t-s)^{\alpha-1} \mathcal{S}(t-s) f_m(s) ds\right\}\right) \leq 2L(A) \int_0^t \zeta(s) ds.$$

3. Main Results

This section will offer our main results for almost periodic waves for the impulsive fractional-order inclusions of type (1). To this end, we will apply the following assumptions related to the proposed approach. Let $D \in \mathcal{P}_b(E)$.

A4. The multivalued function $F(t, y)$ is Weyl almost periodic along t for any $y \in D$, and there exists a function $\beta(t) \in L^1_+(\mathbb{R}, \mathbb{R})$ with

$$r = 4L(A) \int_{-\infty}^t \beta(t) < 1$$

such that $\chi(F(t, D)) \leq \beta(t)\chi(D)$.

A5. There exists a function $\alpha \in L^1_+(\mathbb{R}, \mathbb{R})$ such that

$$\|F(t, y)\| \leq \alpha(t)(1 + \|y\|), \text{ a.e. } t \in \mathbb{R}, y \in E.$$

A6. The sequence $I_k(y)$, $k = \pm 1, \pm 2, \dots$ from continuous compact functions is almost periodic along k for any $y \in D$, and there exist sequences of constants $\{c_k\}$, $\{d_k\}$, $k = \pm 1, \pm 2, \dots$, such that

$$\|I_k(y)\| \leq c_k \|y\| + d_k,$$

where $L(A) \sum_{k=\pm 1, \pm 2, \dots} c_k < 1, \sum_{k=\pm 1, \pm 2, \dots} d_k < \infty$.

A7. There exists a function $\gamma : C[\mathbb{R}, \mathbb{R}_+]$

$$\gamma(t) \geq L(A) \int_{-\infty}^t \alpha(s)(1 + \gamma(s)) ds + L(A) \sum_{t_k < t} (c_k \gamma(t_k) + d_k),$$

where $L(A) > 0$, α, c_k and $d_k, k = \pm 1, \pm 2, \dots$ are from A4–A6.

The following definition will also be useful [58,60].

Definition 2. A function $y \in PC[\mathbb{R}, E]$ is an impulsive mild solution of (1) if

$$y(t) = \int_{-\infty}^t (t-s)^{\alpha-1} \mathcal{S}(t-s) f(s, y(s)) ds + \sum_{t_k < t} \mathcal{S}(t-t_k) I_k(y(t_k)),$$

where $f \in S_{F(\cdot, y(\cdot))}$.

Remark 2. Under hypothesizes and definitions on the paper it follows that, for the almost periodic map $F(t, y)$, there exists [56] a Caratheodory almost periodic function $f(t, y(t)) \in S_{F(., y(.))}$, i.e., $f(t, .)$ is continuous for every $t \in \mathbb{R}$, $f(., y)$ is Borel-measurable for every $y \in E$.

Theorem 1. Under the assumptions A1–A7, there exists at least one almost periodic impulsive mild solution of (1).

Proof. We will consider the set $\mathcal{APC}_b \subset PC[\mathbb{R}, E]$ of all bounded, almost periodic, piecewise continuous functions with point of discontinuity at the moments $t_k, k = \pm 1, \pm 2, \dots$, and we will consider the map $\Theta : \mathcal{APC}_b \rightarrow \mathcal{APC}_b$, such that

$$\Theta(y)(t) = \int_{-\infty}^t (t-s)^{\alpha-1} \mathcal{S}(t-s) f(s, y(s)) ds + \sum_{t_k < t} \mathcal{S}(t-t_k) I_k(y(t_k)). \tag{5}$$

From (5), it can be seen that a fixed point of Θ will be a mild solution of (1).

To prove Theorem 1, we will follow these steps:

Step 1. First, we will show that the right-hand side of (5) is well-defined.

Since, for the function $v(t) = f(t, y(t))$ and sequence $a_k = I_k(y(t_k))$, assumptions A2 and A3 are satisfied, then $v(t)$ and $\{a_k\}$ are bounded, and let

$$\max\{\|v(t)\|_p^{PC}, \|a_k\|\} \leq M_0, M_0 > 0.$$

Now, let $y(t) \in \mathcal{APC}_b$ and $x(t) = \Theta(y)(t)$. Then, in view of Lemma 4, Lemma 3 and the definition of the norm in E_β , we obtain

$$\begin{aligned} \|x(t)\|_\beta &= \int_{-\infty}^t (t-s)^{\alpha-1} \|A^\beta \mathcal{S}(t-s)\| \|v(s)\|_p^{PC} ds + \sum_{t_k < t} \|A^\beta \mathcal{S}(t-t_k)\| \|a_k\| \\ &\leq K_\beta M_0 \left[\alpha \int_{-\infty}^t \int_0^\infty \theta^{1-\beta} \zeta_\alpha(\theta) (t-s)^{-\alpha\beta+\alpha-1} e^{-\lambda\theta(t-s)^\alpha} d\theta ds \right. \\ &\quad \left. + \sum_{t_k < t} \int_0^\infty \theta^{-\beta} \zeta_\alpha(\theta) (t-t_k)^{-\alpha\beta} e^{-\lambda\theta(t-t_k)^\alpha} d\theta \right]. \end{aligned} \tag{6}$$

Then,

$$\begin{aligned} \|x(t)\|_\alpha &\leq K_\beta M_0 \left[\alpha \int_0^\infty \int_0^\infty \theta^{1-\beta} \zeta_\alpha(\theta) \sigma^{-\alpha\beta+\alpha-1} e^{-\lambda\theta\sigma^\alpha} d\theta d\sigma \right. \\ &\quad \left. + \sum_{t_k < t} \int_0^\infty \theta^{-\beta} \zeta_\alpha(\theta) (t-t_k)^{-\alpha\beta} e^{-\lambda\theta(t-t_k)^\alpha} d\theta \right], \end{aligned} \tag{7}$$

where $\sigma = t - s$.

However, we have that

$$\begin{aligned} &\alpha \int_0^\infty \int_0^\infty \theta^{1-\beta} \zeta_\alpha(\theta) \sigma^{-\alpha\beta+\alpha-1} e^{-\lambda\theta\sigma^\alpha} d\theta d\sigma \\ &= \alpha \int_0^\infty \zeta_\alpha(\theta) \int_0^\infty \theta^{1-\beta} \sigma^{-\alpha\beta+\alpha-1} e^{-\lambda\theta\sigma^\alpha} d\sigma d\theta \\ &= \frac{\Gamma(1-\beta)}{\lambda^{1-\beta}}. \end{aligned} \tag{8}$$

On the other hand, by A1, (3) and Lemma 2, for the sum in (7), we obtain

$$\begin{aligned} &\sum_{t_k < t} \int_0^\infty \theta^{-\beta} \zeta_\alpha(\theta) (t-t_k)^{-\alpha\beta} e^{-\lambda\theta(t-t_k)^\alpha} d\theta \\ &= \int_0^\infty \zeta_\alpha(\theta) \left[\sum_{0 < t-t_k \leq 1} (\theta(t-t_k)^\alpha)^{-\beta} e^{-\lambda\theta(t-t_k)^\alpha} \right] \end{aligned}$$

$$\begin{aligned}
 &+ \sum_{j=1}^{\infty} \sum_{j < t - t_k \leq j+1} (\theta(t - t_k)^\alpha)^{-\beta} e^{-\lambda\theta(t - t_k)^\alpha} d\theta \\
 &\leq 2N \left[\frac{1}{\zeta^\beta} + \frac{1}{e^\lambda - 1} \right].
 \end{aligned} \tag{9}$$

Then, from (8), (9) and

$$\Gamma(\alpha)\Gamma(1 - \alpha) \sin \pi\alpha = \pi, \quad 0 < \alpha < 1$$

it follows

$$\|x(t)\|_\alpha \leq K_\beta M_0 \left[\frac{\pi}{\Gamma(\beta) \sin \pi\beta\lambda^{1-\beta}} + 2N \left(\frac{1}{\zeta^\beta} + \frac{1}{e^\beta - 1} \right) \right].$$

Let $\varepsilon > 0$, $\tau \in T$, $q \in Q$, where T and Q are defined in Lemma 2. Then,

$$\begin{aligned}
 \|x(t + \tau) - x(t)\|_\beta &= \|A^\beta(x(t + \tau) - x(t))\| \\
 &\leq \int_{-\infty}^t \|A^\beta \mathcal{S}(t - s)\| \|v(s + \tau) - v(s)\|_p^{PC} ds \\
 &\quad + \sum_{t_k < t} \|A^\beta \mathcal{S}(t - t_k)\| \|a_{k+q} - a_k\| \leq M_\beta \varepsilon,
 \end{aligned}$$

where $|t - t_k| > \varepsilon$ and

$$M_\beta = M_0 \left[\frac{\pi}{\Gamma(\beta) \sin \pi\beta\lambda^{1-\beta}} + 2N \left(\frac{1}{\zeta^\beta} + \frac{1}{e^\beta - 1} \right) \right].$$

The last inequality implies that the function $x(t)$ is almost periodic; hence, \mathcal{APC}_b is not empty.

Step 2. Using ideas from [52], we will prove that, for the map Θ , there exists a fixed point.

Now, in \mathcal{APC}_b , we define the set $\{B_n\} = \bar{c}o\Theta(B_{n-1})$, $n \geq 1$ and, for B_0 , we use the next set

$$B_0 = \{y \in \mathcal{APC}_b, \|y(t)\| < \gamma(t), t \in \mathbb{R}\}, \tag{10}$$

where the function $\gamma(t)$ is from A7.

From Step 1 and the fact that the set \mathcal{APC}_b is constructed of bounded, almost periodic functions, we conclude that B_0 is bounded and it is also convex.

We will prove that $\Theta(B_0) \subset B_0$. Let $y \in B_0$, $x \in \Theta(B_0)$ be such that $x = \Theta(y)$. From A5–A7 and the chain of inequalities

$$\begin{aligned}
 \|x(t)\|_\beta &= \int_{-\infty}^t (t - s)^{\alpha-1} \|A^\beta \mathcal{S}(t - s)\| \|f(s, y(s))\| ds \\
 &\quad + \sum_{t_k < t} \|A^\beta \mathcal{S}(t - t_k)\| \|I_k(y(t_k))\| \\
 &\leq L(A) \left(\int_{-\infty}^t \alpha(t)(1 + \gamma(s)) ds + \sum_{t < t_k} (c_k \gamma(t_k) + d_k) \right) \leq \gamma(t), \quad t \in \mathbb{R},
 \end{aligned}$$

it follows that $x \in B_0$ and $\Theta(B_0) \subset B_0$.

By the definition of $\{B_n\}$ and induction, the sequence $\{B_n\}$ is decreasing and as B_0 is bounded, B_n is also bounded for $n \geq 1$.

Now, we have to prove that the sequence $\{B_n\}$ is equicontinuous at \bar{J}_k , $k = \pm 1, \pm 2, \dots$ and the function $\eta(t)$ is such that

$$\eta(t) = \alpha(t)(1 + H), \quad H = \sup_{t \in \mathbb{R}} \gamma(t).$$

Fix $n \in \mathbb{N}$, $\varepsilon > 0$ and let $y \in B_{n-1}$, $t, t' \in \bar{J}_k$, $k = \pm 1, \pm 2, \dots$. Then, from (5), we obtain

$$\begin{aligned} & \|\Theta(y)(t') - \Theta(y)(t)\| \\ & \leq \int_{-\infty}^{t'} \|(t' - s)^{\alpha-1} \mathcal{I}(t' - s) - (t - s)^{\alpha-1} \mathcal{I}(t - s)\| \|f(s, x(s))\| ds \\ & \quad + \int_t^{t'} \|(t' - s)^{\alpha-1} \mathcal{I}(t' - s)\| \|f(s, y(s))\| ds \\ & \quad + \sum_{i=-\infty}^k \|(t' - t_i)^{\alpha-1} \mathcal{I}(t' - t_i) - (t - t_i)^{\alpha-1} \mathcal{I}(t - t_i)\| \|I_i(y(t_i))\|. \end{aligned} \tag{11}$$

From Lemma 3, it follows that, for any $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$, such that

$$\|(t' - s)^{\alpha-1} \mathcal{I}(t' - s) - (t - s)^{\alpha-1} \mathcal{I}(t - s)\| < \varepsilon,$$

when $|t' - t| < \delta$ and for every measurable set M with a Lebesgue measure $\lambda(M) < \delta$ we have

$$\int_M \eta(s) ds < \varepsilon.$$

Additionally, we have that

$$\sum_{k=\pm 1, \pm 2, \dots} c_k < (L(A))^{-1}, \quad \sum_{k=\pm 1, \pm 2, \dots} d_k < \infty$$

and from (11), we obtain

$$\|\Theta(y)(t') - \Theta(y)(t)\| < \varepsilon L(A) \left(\int_{-\infty}^{t'} \eta(s) ds + \int_t^{t'} \eta(s) ds + \sum_{k=-\infty}^i (c_i H + d_i) \right).$$

The last inequality proves the equicontinuity of the map Θ on J_k , which easily implies the equicontinuity on J_k of the set $B_n|_{J_k} = \overline{\text{co}}(B_{n-1})$. Hence, the set $B_n|_{\bar{J}_k}$ is equicontinuous on \bar{J}_k , $k = \pm 1, \pm 2, \dots$

Then, we need to prove that $\lim_{n \rightarrow \infty} \chi(B_n) = 0$. By means of the invariance of Hausdorff measure of noncompactness and by applying Lemma 1, it follows that for arbitrary $\varepsilon > 0$, there exists a sequence $X = \{x_m\}_{m \in \mathbb{N}}$ such that

$$\chi_{PC}(B_n) \leq 2\chi_{PC}(X) + \varepsilon \leq 2 \sup_{k=\pm 1, \pm 2, \dots} \chi_k(X|_{\bar{J}_k}) + \varepsilon.$$

From the equicontinuity of $B_n|_{\bar{J}_k}$ and nonsingularity of χ , we have $\chi_k(X|_{\bar{J}_k}) = \sup_{t \in \bar{J}_k} \chi(X(t))$, and then

$$\chi_{PC}(B_n) \leq 2 \sup_{k=\pm 1, \pm 2, \dots} \left(\sup_{t \in J_k} \chi(X(t)) + \varepsilon \right) = 2 \sup_{t \in \mathbb{R}} \chi(X(t)) + \varepsilon. \tag{12}$$

Let $Y = \{y_m\}_{m \in \mathbb{N}}$ be a sequence in B_{n-1} such that $x_m = \Theta(y_m)$. Then, from (5), it follows

$$\begin{aligned} \chi_{PC}(B_n) \leq 2 \sup_{t \in \mathbb{R}} \chi \left(\left\{ \int_{-\infty}^t (t-s)^{\alpha-1} \mathcal{I}(t-s) f(s, y_m(s)) ds \right. \right. \\ \left. \left. + \sum_{t_k < t} (t-t_k)^{\alpha-1} \mathcal{I}(t-t_k) I_k(y_m(t_k)) \right\}_{m \in \mathbb{N}} \right) + \varepsilon. \end{aligned}$$

On the other hand, from A6, we obtain

$$\chi \left(\sum_{t_k < t} (t-t_k)^{\alpha-1} \mathcal{I}(t-t_k) I_k(\{y_m(t_k)\}_{n \in \mathbb{N}}) \right)$$

$$\leq \sum_{t_k < t} \chi((t - t_k)^{\alpha-1} \mathcal{I}(t - t_k) I_k(\{y_m(t_k)\}_{m \in \mathbb{N}})) = 0. \tag{13}$$

As $y_m \in B_{n-1} \subset B_0$, $m \in \mathbb{N}$, we have that

$$\sup_{m \in \mathbb{N}} \|f(t, y_m(t))\| \leq \sup_{m \in \mathbb{N}} \alpha(t)(1 + \|y_m(t)\|) \leq \eta(t),$$

a.e., $t \in \mathbb{R}$. Then, by A1 and monotonicity of χ we get

$$\chi(\{f(t, y_m(t))\}_{m \in \mathbb{N}}) \leq \beta(t)\chi(\{y_m(t)\}_{m \in \mathbb{N}}) \leq \beta(t)\chi_{PC}(B_{n-1}),$$

a.e., $t \in \mathbb{R}$ and from Lemma 4 it follows

$$\begin{aligned} & \chi\left(\left\{\int_{-\infty}^t (t-s)^{\alpha-1} \mathcal{I}(t-s) f(s, y_m(s)) ds\right\}_{m \in \mathbb{N}}\right) \\ & \leq 2L(A) \int_{-\infty}^t \beta(s)\chi_{PC}(B_{n-1}) ds. \end{aligned} \tag{14}$$

Then, from (13) and (14) for (12), we have

$$\chi_{PC}(B_n) \leq \varepsilon + 4L(A)\chi_{PC}(B_{n-1}) \int_{-\infty}^t \beta(s) ds,$$

and $\chi_{PC}(B_n) \leq r\chi_{PC}(B_{n-1})$, where r is from A4.

Hence,

$$0 \leq \chi_{PC}(B_n) \leq r^{n-1}\chi_{PC}(B_1), \text{ and } \lim_{n \rightarrow \infty} \chi_{PC}(B_n) = 0.$$

Then, from the generalized Cantor’s intersection property for the Hausdorff measure of noncompactness χ_{PC} , it follows that the set $B = \bigcap_1^\infty B_n$ is not an empty, convex and compact subset of \mathcal{APC}_b .

On the other hand, from $\Theta(B) \subset \Theta(B_n) \subset \overline{c\partial}(\Theta(B_n)) = B_{n+1}$ for every $n \in \bigcap_{n=2}^\infty B_n$ and, from the $B_n \subset B_1$, for every $n \in \mathbb{N}$, we have $\Theta(B) \subset B$.

The last step is to prove that $\Theta : B \rightarrow B$ is a continuous map. Indeed, let the sequence $\{y_n\}_{n \in \mathbb{N}} \subset B$ and let $y_n \rightarrow y$, $y \in B$ on \mathcal{APC}_b . Then from (5) for $t \in \mathbb{R}$ we have

$$\begin{aligned} \|\Theta(y_n)(t) - \Theta(y)(t)\| & \leq \left\| \int_{-\infty}^t (t-s)^{\alpha-1} \mathcal{I}(t-s) [f(s, y_n(s)) - f(s, y(s))] ds \right\| \\ & \quad + \left\| \sum_{t_k < t} (t-t_k)^{\alpha-1} \mathcal{I}(t-t_k) [I_k(y_n(t_k)) - I_k(y(t_k))] \right\| \\ & \leq L(A) \left\{ \int_{-\infty}^t \|f(s, y_n(s)) - f(s, y(s))\| ds \right. \\ & \quad \left. + \sum_{t_k < t} \|I_k(y_n(t_k)) - I_k(y(t_k))\| \right\}. \end{aligned} \tag{15}$$

By the continuity of $f(s, \cdot)$, we have $\lim_{n \rightarrow \infty} \|f(s, y_n(s)) - f(s, y(s))\| = 0$, and then A2 implies $\|f(s, y_n(s)) - f(s, y(s))\| \leq 2\eta(s)$.

From the dominated convergence theorem and from the last inequality we get

$$\lim_{n \rightarrow \infty} \int_{-\infty}^t \|f(s, y_n(s)) - f(s, y(s))\| ds = 0. \tag{16}$$

Similarly, A3 implies that

$$\lim_{n \rightarrow \infty} \sum_{t_k < t} \|I_k(y_n(t_k)) - I_k(y(t_k))\| = 0. \tag{17}$$

Hence, by (16) and (17) we obtain

$$\lim_{n \rightarrow \infty} \|\Theta(y_n) - \Theta(y)\| = 0.$$

Therefore, the map $\Theta : B \rightarrow B$ verifies the hypotheses of the Schauder fixed-point theorem.

The proof of Theorem 1 is complete. \square

Remark 3. The result in Theorem 1 generalizes some results of impulsive mild solutions for semilinear differential inclusions in [51,52], considering fractional-order derivatives. For $\alpha = 1$, the results in [52] can be considered as corollaries of the proposed new results. We also generalize the results of almost periodic waves for fractional-order inclusions for the impulsive case.

Remark 4. The existence criteria provided by Theorem 1 also generalize the results in [11–13] considering impulsive perturbations, which is more natural and realistic, and, therefore, the new results offer an extended horizon for applications. It is worth noting that, if the inclusion (1) is without impulsive perturbations at some instants or the impulsive function $I_k(\cdot) = 0$, $k = \pm 1, \pm 2, \dots$, then some existence criteria in [11–13] can be obtained as corollaries from our result.

Remark 5. Since the operator A and multi-function F generalize many specific cases, it is clear that Theorem 1 can be applied to a number of situations, including the special cases, when A is the Laplace operator, which is very useful in numerous fields.

4. Application to GRNs

Let $\alpha, 0 < \alpha < 1$. In order to apply our results, we will consider the class of the following fractional-order impulsive GRN inclusions given by

$$\begin{cases} {}^c D^\alpha m_i(t) \in -b_i m_i(t) + \sum_{j=1}^n w_{ij}(t) g_j(p_j(t)) + m_i^u(t), t \neq t_k, \\ {}^c D^\alpha p_i(t) \in -c_i p_i(t) + d_i(t) m_i(t), t \neq t_k, \\ \Delta m_i(t_k) = I_{ik}^1(m_i(t_k)), \\ \Delta p_i(t_k) = I_{ik}^2(p_i(t_k)), \end{cases} \tag{18}$$

that regulates the concentrations of mRNA $m_i(t)$ and protein $p_i(t)$ at time t , where:

(i) $i, i = 1, 2, \dots, n$ is the number of the node, the positive constants b_i, c_i represent the dilution rates, using the dimensionless transcriptional bounded rate $u_{ij}(t)$ at time t of transcription factor j to i , $w_{ij}(t)$ are defined as

$$w_{ij}(t) = \begin{cases} u_{ij}(t) & \text{when } j \text{ is an activator of gene } i, \\ -u_{ij}(t) & \text{when } j \text{ is a repressor of gene } i, \\ 0 & \text{when there is no link from the node } j \text{ to gene } i, \end{cases}$$

$d_i(t) \in \mathbb{R}$ denotes the translation rate; g_j represents the regulatory of the protein function and is in the form

$$g_j(p_j) = \frac{(p_j/\beta_j)^{H_j}}{1 + (p_j/\beta_j)^{H_j}},$$

where H_j denotes the Hill coefficients and β_j are positive scalars, $m_i^u(t)$ is defined as $m_i^u(t) = \sum_{j \in R_i} u_{ij}(t)$, where R_i is the set of all repressors of gene i ;

(ii) the impulsive instants $t_k, k = \pm 1, \pm 2, \dots$, are such that $t_k \in \mathcal{T} \cap [0, 100]$ and satisfy A1;

(iii) the scalars $m_i(t_k) = m_i(t_k^-)$ and $p_i(t_k) = p_i(t_k^-)$ represent the concentration of mRNA $m_i(t)$ and protein $p_i(t)$ before an impulsive perturbation at time t_k , respectively, $m_i(t_k^+)$ and $p_i(t_k^+)$ are the levels in the concentration of mRNA $m_i(t)$ and protein $p_i(t)$ after an impulsive perturbation at the moment t_k , respectively, the impulsive functions I_{ik}^1, I_{ik}^2 describe the abrupt changes in $m_i(t)$ and $p_i(t)$ at the impulsive moments t_k , $i = 1, 2, \dots, n, k = \pm 1, \pm 2, \dots$.

Due to their importance in biology, neuroscience, medicine, etc., GRNs have been widely studied for $\alpha = 1$ and some interesting results have been obtained, as shown in references [62–66], including impulsive GRNs [67–69].

In addition, recently, fractional-order derivatives have been involved in the GRN models in [70–72] and impulses have been considered in fractional-order models very recently [73].

If we denote $y = (m, p) = (m_1, m_2, \dots, m_n, p_1, p_2, \dots, p_n)$,

$$B = \text{diag}(b_1, \dots, b_n), C = \text{diag}(c_1, \dots, c_n),$$

$$G^p(p(t)) = (G_1^p(p(t)), \dots, G_n^p(p(t)))^T, \quad G_i^p(p(t)) = \sum_{j=1}^n w_{ij}(t)g_j(p_j(t)) + m_i^u(t),$$

$$G^m(m(t)) = (d_1(t)m_1(t), \dots, d_n(t)m_n(t))^T,$$

$$I_k^1 = \text{diag}(I_{1k}^1, \dots, I_{nk}^1), \quad I_k^2 = \text{diag}(I_{1k}^2, \dots, I_{nk}^2),$$

$$-A = \begin{bmatrix} -B & 0 \\ 0 & -C \end{bmatrix}, \quad G(t, y) = \begin{bmatrix} G^p(p(t)) \\ G^m(m(t)) \end{bmatrix}, \quad I_k(y(t_k)) = \begin{bmatrix} I_k^1(m(t_k)) \\ I_k^2(p(t_k)) \end{bmatrix},$$

then (18) can be represented by (1) with $E = \mathbb{R}^{2n}, v(z) = G(t, y(z))$ for $z \in [0, 100]$ and $a_k = I_k(y(t_k)), k = \pm 1, \pm 2, \dots$

If we assume that:

- (a) the function $v(z)$ is almost periodic in the sense of Weyl;
- (b) The sequence $\{a_k\}, a_k \in \mathbb{R}^{2n}, k = \pm 1, \pm 2, \dots$ is almost periodic; then, conditions A2 and A3 are met.

Note that, condition (a) is satisfied when the functions $d_i(t), w_{ij}(t), m_i^u(t)$ are almost periodic in the sense of Weyl piecewise functions for any $i, j = 1, 2, \dots, n$.

The conditions of Lemma 3 are satisfied for

$$\lambda = \min\{b_1, \dots, b_n, c_1, \dots, c_n\},$$

considering e^{-At} as the analytic semigroup of the operator A, and

$$A^{-\beta} = \frac{1}{\Gamma(\beta)} \int_0^\infty t^{\beta-1} e^{-At} dt.$$

In addition, if we suppose that A4–A7 are all satisfied for (18), then, using Theorem 1, we can conclude that there exists at least one almost periodic impulsive mild solution of (18).

Remark 6. In the above example, we generalize the GRN models proposed in [62–69] to the case of fractional-order inclusions.

Remark 7. We note that the impulsive perturbations considered in (18) can be applied as an efficient control strategy. Hence, the presented examples not only show the effectiveness of the proposed theoretical results, but also illustrate how the behavior of a class of fractional GRN inclusions can be controlled via appropriate impulsive perturbations.

5. Conclusions

In this paper, we extend the concept of almost periodic waves to fractional-order inclusions under impulsive perturbations. Using the theory of an operators' semigroup, Hausdorff measure of noncompactness, fixed-point theorems and the techniques based on fractional calculus, sufficient conditions for existence and uniqueness are established. The proposed results are illustrated on a fractional-order impulsive GRN model. Since the almost periodicity is much more general than pure periodicity and more appropriate for applications, our results are meaningful and valuable for the development of the theory of impulsive fractional-order inclusions. The ideas and approaches given in this paper can be extended and applied to problems including different types of delays. Almost periodicity is closely related to the boundedness of the solutions. Hence, more interesting topics, including qualitative properties such as stability, stabilization, synchronization, asymptotic properties, as well as applications, can be developed using the proposed results.

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