

## GUARDING TERRAINS VIA LOCAL SEARCH

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ABSTRACT. We obtain a polynomial time approximation scheme for the 1.5D terrain guarding problem, improving upon several recent constant factor approximations. Our algorithm is a local search algorithm inspired by the recent results of Chan and Har-Peled [3] and Mustafa and Ray [18]. Our key contribution is to show the existence of a planar graph that appropriately relates the local and global optimum.

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### 1 Introduction

A *1.5D terrain* is a polygonal chain in the plane that is  $x$ -monotone, that is, any vertical line intersects the chain at most once. A terrain  $T$  consists of a set of  $m$  vertices  $\{v_1, v_2, \dots, v_m\}$ . The vertices are ordered in increasing order with respect to their  $x$ -coordinates. There is an edge connecting  $v_i$  with  $v_{i+1}$  for all  $i = 1, 2, \dots, m - 1$ . For any two points  $a, b \in T$ , we say that  $a$  sees  $b$  if each point in  $\overline{ab}$  lies above or on the terrain.

In this paper, we consider the *discrete terrain guarding problem* in which we are given a terrain  $T$  and finite sets  $X, G \subseteq T$ . For a set  $G' \subseteq G$ , we say that  $G'$  covers/sees/guards  $X$  if every point in  $X$  can be seen by at least one point in  $G'$ . The goal is to find a minimum cardinality subset of  $G$  that covers  $X$ .

A closely related problem is the *art gallery problem*. Again the goal is to find a minimum cardinality guarding set, but in this setting we must guard a simple polygon. The basic version of this problem, *vertex guarding*, allows guards to be placed only at the vertices of the polygon. Another version, *point guarding*, allows guards to be placed anywhere inside the polygon.

We note that our discrete terrain guarding problem can be reduced to a version of the art gallery problem. To do this, we pick a point  $v_0$  above the terrain such that  $\langle v_0, v_1, \dots, v_m \rangle$  is the sequence of vertices of a simple polygon in counter-clockwise order. For this, it suffices to pick  $v_0$  with sufficiently high  $y$ -coordinate. The resulting polygon will in fact be  $x$ -monotone, and our discrete terrain guarding problem reduces to that of finding a minimum cardinality subset of  $G$  that guards  $X$  in this polygon.

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The vertex guarding version of the art gallery problem for simple polygons was shown to be NP-complete by Lee and Lin [17] and was later shown to be APX-hard by Eidenbenz et al. [8]. This means that there is an  $\epsilon > 0$  such that no polynomial time algorithm can compute a guarding set whose cardinality is within a  $(1 + \epsilon)$  factor of the cardinality of an optimal guarding set, unless  $P = NP$ . The papers [17, 8] also establish the same hardness results for point guarding. Ghosh gives an  $O(\log n)$ -approximation algorithm for vertex guarding an  $n$ -vertex simple polygon [12]. This was the best approximation guarantee known for this problem for a long time. Very recently, King and Kirkpatrick [14] succeeded in giving an  $O(\log \log \text{OPT})$ -approximation, where  $\text{OPT}$  is the size of the optimal solution. The point guarding problem seems to be much more difficult as not as much is known about it [6]. A constant factor approximation is given by Krohn and Nilsson for the special case of the problem when the polygon is  $x$ -monotone [16]. They also give an approximation algorithm for rectilinear polygons that returns a solution with  $O(\text{OPT}^2)$  guards.

**Previous Work.** Most of the past research on discrete terrain guarding has focused on developing approximation algorithms. The first constant factor approximation was a combinatorial algorithm given by Ben-Moshe et al. [2]. Clarkson and Varadarajan [5] also give a constant factor approximation based on rounding a linear programming relaxation. King gave a simple combinatorial 4-approximation which was later determined to actually be a 5-approximation [13]. Recently, Elbassioni et al. [9] gave a 4-approximation that also works for a weighted version of the problem.

All of the approximation algorithms use the following “order claim”:

**Claim 1** *Let  $a, b, c, d$  be four points on the terrain in increasing order according to  $x$ -coordinate. If  $a$  sees  $c$  and  $b$  sees  $d$  then  $a$  sees  $d$ .*

Chen et al. [4] claimed that terrain guarding is NP-hard, but the proof was never completed formally. Natural attempts at constructing NP-hardness reductions for the terrain guarding problem do not work because of the order claim. Recently King and Krohn [15] were able to get around the order claim and prove NP-hardness. Since the objective function is the number of guards, which is an integer upper bounded by the input size, this NP-hardness result also rules out a *Fully Polynomial-Time Approximation Scheme (FPTAS)*. An FPTAS is an algorithm that, for any parameter  $\epsilon > 0$ , runs in time that is polynomial in the input size and  $1/\epsilon$ , and returns a solution that with at most  $(1 + \epsilon)$  times the optimal number of guards.

**Our Contribution.** We give a polynomial time algorithm that returns a guard cover whose cardinality is at most  $(1 + \epsilon) \cdot \text{OPT}$  for any  $\epsilon > 0$ . Here,  $\text{OPT}$  denotes the cardinality of an optimal guard cover. Thus we obtain the first PTAS for the problem improving upon several recent constant factor approximations. Given the hardness result [15], this settles the computational complexity of the problem.

The inspiration for our work comes from the recent results of Chan and Har-Peled [3] and Mustafa and Ray [18]. Chan and Har-Peled show that a local search algorithm actually

yields a PTAS for the maximum independent set problem given a collection of disks. Unlike a previous PTAS for the problem [10], their analysis does not use packing arguments and thus also applies to “pseudo-disks”. Mustafa and Ray consider several geometric hitting set and set cover problems and describe local search algorithms that yield PTASs. For instance, in a rather surprising result they obtain a PTAS for the problem of covering a set of points by the smallest subset of a given set of disks. Both papers use separator theorems for planar graphs. In particular, they show that there exists a planar graph that relates the locally optimal solution returned by the local search and the global optimal solution. The separator theorem is then used to show that the locally optimal solution is not much worse than the global optimum.

Our PTAS for the terrain guarding problem is also based on local search. Our key contribution is to show the existence of an appropriate planar graph for the terrain guarding context. Having shown this, the rest of the analysis is very similar to that of Mustafa and Ray [18]. For completeness, this analysis is reviewed in Section 2.

## 2 Guarding Terrains via Local Search

Recall that our input is a polygonal terrain, a set  $X$  of points on the terrain that need to be guarded, a set  $G$  of possible guard locations, and a parameter  $0 < \epsilon < 1$ . For purposes of exposition, we will initially assume that  $X \cap G = \emptyset$ . We later show how this assumption can be removed. We describe a polynomial time algorithm that returns a subset  $Q \subseteq G$  that sees  $X$ , so that  $|Q|$  is at most a factor  $(1 + \epsilon)$  times the size of the smallest subset of  $G$  that sees  $X$ . We may assume that  $G$  covers  $X$ . Let  $n$  denote the input size – the number of vertices in the terrain, plus  $|G|$ , plus  $|X|$ .

### 2.1 The Algorithm

We say that a subset of  $G$  that sees  $X$  is  $b$ -locally optimal if one cannot obtain a smaller set of guards that sees  $X$  by deleting at most  $b$  guards and inserting fewer guards. Our algorithm simply returns a  $b$ -locally optimal solution for  $b = \frac{\alpha}{\epsilon^2}$ , where  $\alpha$  is a suitably large constant. It does this using the following local search algorithm:

1. Initialize  $Q$  to an arbitrary subset of  $G$  that covers  $X$ .
2. Determine if there exists subset  $S \subseteq Q$  of size at most  $b$  and subset  $S' \subseteq G \setminus Q$  of size at most  $|S| - 1$  such that  $(Q \setminus S) \cup S'$  guards  $X$ .
3. If such  $S$  and  $S'$  exist, set  $Q \leftarrow (Q \setminus S) \cup S'$ , and go back to Step 2. If such  $S$  and  $S'$  do not exist, return  $Q$ .

Notice that upon termination  $Q$  is  $b$ -locally optimal. Furthermore, an update to  $Q$  in Step 3 reduces the size of  $|Q|$  by at least one. This means that at most  $n$  such updates happen. In Step 2, our algorithm looks at all possible choices for subsets  $S$  and  $S'$ . The number of choices for  $S$  (resp.  $S'$ ) is bounded by  $\sum_{i=0}^b \binom{n}{i}$ . Thus the running time of a

single execution of Step 2 is  $n^{O(b)} = O(1/\epsilon^2)$ . Since Steps 2 and 3 are executed at most  $n$  times, the running time of the entire algorithm is also  $n^{O(1/\epsilon^2)}$ .

## 2.2 Approximation Analysis

Let  $R'$  denote the optimal cover for  $X$ , and  $B'$  the set  $Q$  of guards output by our local search algorithm upon termination. We show that  $|B' \setminus R'| \leq (1 + \epsilon)|R' \setminus B'|$ . Note that this implies  $|B'| \leq (1 + \epsilon)|R'|$ , as desired. Let  $R \equiv R' \setminus B'$  and  $B \equiv B' \setminus R'$ . We remove from  $X$  all the points seen by  $R' \cap B'$ , and abusing notation, we denote the resulting set by  $X$ . So now both  $R$  and  $B$  cover  $X$  and we wish to show that  $|B| \leq (1 + \epsilon)|R|$ . Note that  $B$  is a  $b$ -locally optimal cover for (the new)  $X$ . We will refer to points in  $B$  as blue points and points in  $R$  as red points.

The following lemma is our main contribution; it shows that the *locality condition* of Mustafa and Ray [18] is satisfied.

**Lemma 2** *There exists a planar graph  $\mathcal{G} = (V \equiv R \cup B, E)$  with the property that for each  $x \in X$ , there is an edge  $(r, b)$  in  $\mathcal{G}$  between guards  $r \in R$  and  $b \in B$  that both see  $x$ .*

We will prove this lemma in the following section. In the remainder of this section, we show how the lemma implies that  $|B| \leq (1 + \epsilon)|R|$ ; this is similar to [3, 18]. We need the following partition theorem on planar graphs due to Frederickson [11]. For  $U \subseteq V$ , let  $\Gamma(U)$  denote the set of neighbors in  $\mathcal{G}$  of vertices in  $U$  with  $U$  excluded. That is,

$$\Gamma(U) = \{v \in V \setminus U \mid \exists u \in U \text{ with } (u, v) \in E\}.$$

Let  $\mu = |V|$ .

**Lemma 3** *For any parameter  $1 \leq r \leq \mu$ , we can find a set  $S \subseteq V$  of size at most  $c_1\mu/\sqrt{r}$  and a partition of  $V \setminus S$  into  $\mu/r$  sets  $V_1, V_2, \dots, V_{\mu/r}$ , satisfying (i)  $|V_i| \leq c_2r$ , (ii)  $|\Gamma(V_i)| \leq c_3\sqrt{r}$ , and (iii)  $\Gamma(V_i) \cap V_j = \emptyset$  for  $i \neq j$ . Here,  $c_1, c_2$ , and  $c_3$  are absolute positive constants.*

Note that condition (iii) implies that  $\Gamma(V_i) \subseteq S$  for each  $i$ . Let us apply the lemma with  $r \equiv b/c_2$ . Let  $R_i = R \cap V_i$  and  $B_i = B \cap V_i$ . Note that  $|B_i| \leq |V_i| \leq c_2r = b$ .

We now claim  $(B \setminus \{B_i\}) \cup (R_i \cup \Gamma(V_i))$  guards  $X$ . To see this, consider a point  $x \in X$ . If some guard in  $B \setminus \{B_i\}$  sees  $x$ , then this guard is also in  $(B \setminus \{B_i\}) \cup (R_i \cup \Gamma(V_i))$ . Suppose therefore that  $x$  is not seen by any guard in  $B \setminus \{B_i\}$ , and it is seen by one or more guards in  $B_i$ . Then by the property of the planar graph  $\mathcal{G}$  in Lemma 2, there is a  $y \in B_i$  and a  $z \in R$  such that both  $y$  and  $z$  see  $x$ , and  $(y, z)$  is an edge in  $\mathcal{G}$ . Since  $y \in B_i \subseteq V_i$ ,  $z$  must belong to  $V_i \cup \Gamma(V_i)$  by condition (iii) of Lemma 3. Since  $z \in R$ , this means  $z \in R_i \cup \Gamma(V_i)$ . Thus,  $R_i \cup \Gamma(V_i)$  sees  $x$ . We conclude that  $(B \setminus \{B_i\}) \cup (R_i \cup \Gamma(V_i))$  guards  $X$ .

Since  $B$  is  $b$ -locally optimal for  $X$ , and  $|B_i| \leq b$ , we claim that

$$|B_i| \leq |R_i| + |\Gamma(V_i)|. \tag{1}$$

For otherwise,  $(B \setminus \{B_i\}) \cup (R_i \cup \Gamma(V_i))$  guards  $X$  and is smaller than  $B$ , contradicting the fact that  $B$  is  $b$ -locally optimal.

The reader should interpret Inequality (1) as saying that in each “piece”  $V_i$ , the number of blue guards is not much more than the number of red guards. From this, we now derive that in the entire graph  $\mathcal{G}$ , the number of blue guards is not much more than the number of red guards. For this, we will use Lemma 3, which in turn is based on the planarity of  $\mathcal{G}$ . In this way, the locality condition allows us to derive global near-optimality of  $B$  from local optimality.

We have

$$\begin{aligned} |B| &\leq |S| + \sum_i |B_i| \\ &\leq |S| + \sum_i |R_i| + \sum_i |\Gamma(V_i)| \\ &\leq \sum_i |R_i| + c \frac{\mu}{\sqrt{r}} \\ &\leq |R| + c \frac{\mu}{\sqrt{r}} \\ &\leq |R| + c' \frac{|R| + |B|}{\sqrt{b}}, \end{aligned}$$

where  $c$  and  $c'$  are positive constants. In this sequence of inequalities, the first follows because each element of  $B$  belongs either to  $S$  or some  $B_i$ ; the second follows from Inequality (1); the third from the upper bound on the size of  $S$  and each of the  $\Gamma(V_i)$ 's, and the number of  $V_i$ 's; the fourth follows because the  $R_i$  are disjoint subsets of  $R$ ; and the fifth because  $r = b/c_2$ .

Rearranging, we get

$$|B| \leq |R| \left( \frac{1 + c'/\sqrt{b}}{1 - c'/\sqrt{b}} \right).$$

We pick  $b = \frac{9(c')^2}{\epsilon^2}$ , so that  $c'/\sqrt{b} = \epsilon/3$ . We have

$$|B| \leq |R| \left( \frac{1 + \epsilon/3}{1 - \epsilon/3} \right) \leq (1 + \epsilon)|R|$$

for  $\epsilon \leq 1$ .

### 2.3 Proof of Lemma 2

We begin with some notation. For points  $a$  and  $b$  on the terrain, we say  $a \leq b$  to mean that the  $x$ -coordinate  $a.x$  of  $a$  is at most  $b.x$ . We use the notation of intervals that this implies – for instance,  $[a, b]$  denotes all points  $c$  on the terrain so that  $a \leq c \leq b$ .

We now prove Lemma 2. We first construct the planar graph  $\mathcal{G}$  and then show that it has the desired properties. For each  $x \in X$ , let  $\lambda(x)$  denote the leftmost point that sees  $x$

among points in  $R \cup B$  to the left of  $x$ , assuming such a point does exist. Similarly, let  $\rho(x)$  denote the rightmost point that sees  $x$  among points in  $R \cup B$  to the right of  $x$ , assuming such a point does exist. Note that at least one of  $\lambda(x)$  or  $\rho(x)$  does exist.

Let  $A_1$  denote the set of segments  $\overline{\lambda(x)x}$ , for  $x \in X$ . Because of the order claim, these segments do not cross. For each  $v \in R \cup B$ , shoot a vertical ray up from  $v$ ; if this ray hits some segment in  $A_1$ , let  $\overline{\lambda(y)y}$  denote the first such segment hit; we add the edge  $(v, \lambda(y))$  to a set  $E_1$  if  $v$  and  $\lambda(y)$  are of different colors.

Now, the edges in  $A_1 \cup E_1$  can be embedded above the terrain in a non-crossing way. To see this, let  $A_1$  be embedded as the original straight line segments. To embed an edge of the form  $(v, \lambda(y)) \in E_1$  as above, we travel straight up from  $v$  till we hit  $\overline{\lambda(y)y}$ , and then slide along the segment  $\overline{\lambda(y)y}$  to reach  $\lambda(y)$ . See Figure 1. A more formal argument that  $A_1 \cup E_1$  can be so embedded is given next.

### 2.3.1 Planarity: Detail

We present here a formal argument for a piece of Lemma 2 that shows that the graph  $(X \cup R \cup B, A_1 \cup E_1)$  has a planar embedding with  $X \cup R \cup B$  on a horizontal line and  $A_1 \cup E_1$  drawn above the line. We now think of  $A_1$  as a set of combinatorial edges rather than as line segments. Let us say that two edges in  $A_1 \cup E_1$  *cross* if all four end points are distinct, and can be ordered as  $a < b < c < d$  with the edges being  $(a, c)$  and  $(b, d)$ . Notice that crossing in this sense is determined entirely by the ordering of the endpoints, and makes no reference to a drawing of the edges. We first argue that no two edges in  $A_1 \cup E_1$  cross.

1. Let  $(\lambda(x), x)$  and  $(\lambda(y), y)$  be any two edges in  $A_1$ . They do not cross, for if  $\lambda(x) < \lambda(y) < x < y$ , then by the order claim,  $y$  sees  $\lambda(x)$ , contradicting the definition of  $\lambda(y)$ .
2. We argue that an edge in  $A_1$  and an edge in  $E_1$  do not cross. Let us recall how edges in  $E_1$  are defined. Let us say that an edge  $(\lambda(x), x)$  is *above*  $v \in R \cup B$  if  $\lambda(x) < v < x$ . We look at all edges in  $A_1$  that are above  $v$ , and if this set is non-empty we find the “innermost” such edge  $(\lambda(y), y)$ . We add the edge  $(\lambda(y), v)$  to  $E_1$  if  $\lambda(y)$  and  $v$  are of opposite colors. Let us say that  $(\lambda(y), y) \in A_1$  *defines*  $(\lambda(y), v)$  in this case.

Now suppose  $(\lambda(y), v)$  crosses some  $(\lambda(x), x) \in A_1$ . Now if  $\lambda(x) < \lambda(y) < x < v$ , then  $(\lambda(x), x)$  and  $(\lambda(y), y)$  cross, a contradiction. On the other hand, if  $\lambda(y) < \lambda(x) < v < x$ , there are two cases: either  $x \leq y$ , in which case the contradiction is that  $(\lambda(y), y)$  is not the innermost edge in  $A_1$  that is above  $v$ ; or  $x > y$ , in which case the contradiction is that  $(\lambda(y), y)$  and  $(\lambda(x), x)$ , which are both edges in  $A_1$ , cross.

3. We now show that no two edges in  $E_1$  cross. Consider two such edges  $(\lambda(y_1), v_1)$  and  $(\lambda(y_2), v_2)$ , defined by  $(\lambda(y_1), y_1)$  and  $(\lambda(y_2), y_2)$  respectively. These edges do not cross, for if  $\lambda(y_1) < \lambda(y_2) < v_1 < v_2$ , then  $(\lambda(y_1), v_1)$  crosses  $(\lambda(y_2), y_2)$ , which contradicts the fact that an edge in  $E_1$  and an edge in  $A_1$  do not cross.

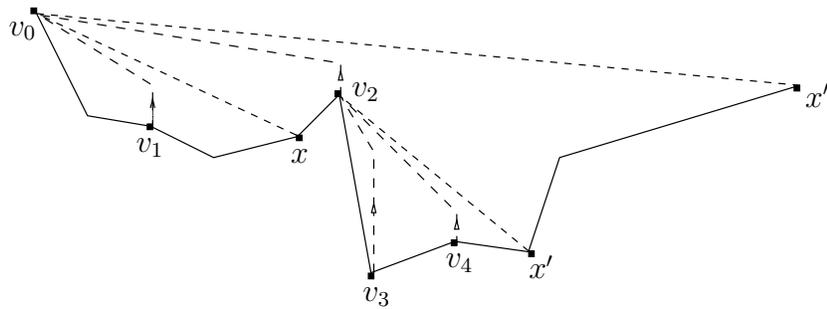


Figure 1: The embedding of  $A_1 \cup E_1$ , with  $X = \{x, x', x''\}$ , and  $R \cup B = \{v_0, v_1, v_2, v_3, v_4\}$ . Segments in  $A_1$  are shown in dashed lines, and the edges in  $E_1$  are embedded as dashed curves with arrows. Note that  $v_0 = \lambda(x) = \lambda(x'')$ , and  $v_2 = \lambda(x')$ .

Thus, no two edges in  $A_1 \cup E_1$  cross. From this, it follows that the required embedding exists. For instance, embed the vertices  $X \cup B \cup R$  as distinct points in the correct order on the lower half of the unit circle, and draw the edges in  $A_1 \cup E_1$  as straight line segments. Using convexity arguments, this can be seen to be a planar embedding. Now we “bend” the lower half of the unit circle into a horizontal segment, allowing the drawing of edges in  $A_1 \cup E_1$  to now become curved.

### 2.3.2 Remainder of Graph $\mathcal{G}$

Let  $A_2$  denote the set of segments  $\overline{x\rho(x)}$ , for  $x \in X$ . Again, these segments do not cross. For each  $v \in R \cup B$ , shoot a vertical ray up from  $v$ ; if this ray hits some segment in  $A_2$ , let  $\overline{y\rho(y)}$  denote the first such segment hit; we add the edge  $(v, \rho(y))$  to a set  $E_2$  if  $v$  and  $\rho(y)$  are of opposite colors.

The edges in  $A_2 \cup E_2$  can also be embedded above the terrain in a non-crossing way. We “flip” the embedding of  $A_1 \cup E_1$  to obtain a non-crossing embedding below the terrain; see Figure 2. This gives us a planar embedding of  $A_1 \cup E_1 \cup A_2 \cup E_2$ .

Finally, for each  $x \in X$ , we add the edge  $(\lambda(x), \rho(x))$  to a set  $E_3$  if  $\lambda(x)$  and  $\rho(x)$  are of different colors. Our graph  $\mathcal{G}$  consists of the edge set  $E_1 \cup E_2 \cup E_3$ . This is a planar graph; just embed  $E_1$  and  $\overline{E_2}$  as above, and for each  $(\lambda(x), \rho(x)) \in E_3$ , embed it using the embedding of the segments  $\overline{\lambda(x)x}$  and  $\overline{x\rho(x)}$ .

### 2.3.3 $\mathcal{G}$ witnesses $X$

Now we need to show that for each  $x \in X$ , there are points  $r \in R$  and  $b \in B$  that see  $x$ , and  $(r, b) \in E_1 \cup E_2 \cup E_3$ . Fix an  $x \in X$ . If  $\lambda(x)$  and  $\rho(x)$  are of different colors, then  $(\lambda(x), \rho(x)) \in E_3$ , and we are done. Otherwise, it must be the case that there are red and blue points to the left of  $x$  that see  $x$ , or that there are red and blue points to the right of  $x$  that see  $x$ .

Let us assume that the first case holds (there are red and blue points to the left

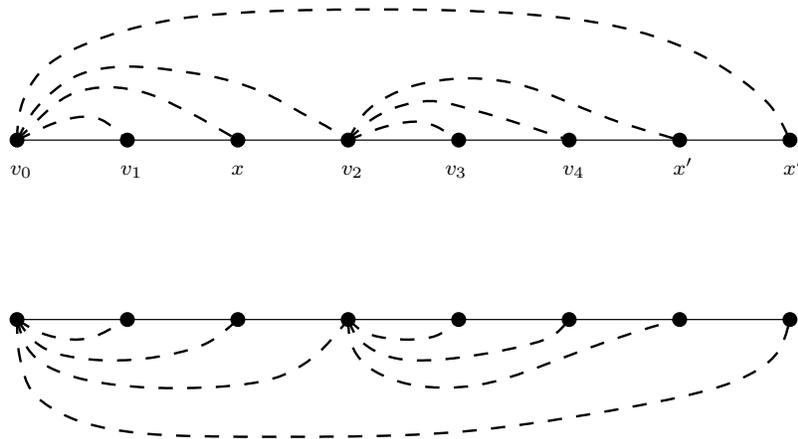


Figure 2: A combinatorial embedding of  $A_1 \cup E_1$  from Figure 1, and flipping it so that  $A_1 \cup E_1$  is now embedded below the terrain. Note that only the edges in  $A_1 \cup E_1$  are being flipped to make room for  $A_2 \cup E_2$ ; the vertex set  $R \cup B \cup X$  retains its embedding.

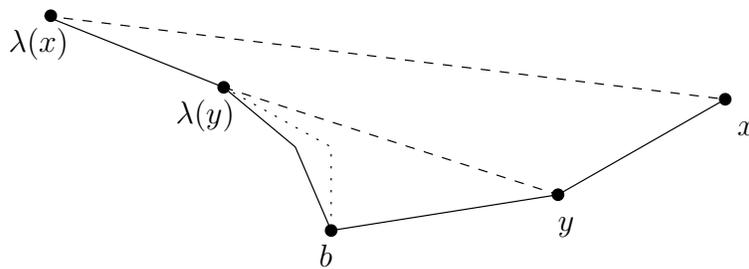


Figure 3: Here,  $\lambda(y)$  sees  $y$  and  $b$  sees  $x$ . By the order claim,  $\lambda(y)$  sees  $x$ .

of  $x$  that see  $x$ ), and that  $\lambda(x)$  is red. The other situations are symmetric. Let  $b$  be the leftmost blue point that sees  $x$ ; it must be that  $b \in (\lambda(x), x)$ . Thus the ray shot up from  $b$  hits  $\overline{\lambda(x)x}$ ; let  $\overline{\lambda(y)y}$  be the first segment in  $A_1$  that it hits. See Figure 3. Because segments in  $A_1$  don't cross, it must be that  $\lambda(y) \in [\lambda(x), b)$  and  $y \in (b, x)$ . The order claim (applied to  $\lambda(y), b, y$ , and  $x$ ) implies that  $\lambda(y)$  sees  $x$ . Now  $\lambda(y)$  cannot be blue, otherwise  $b$  is not the leftmost blue point that sees  $x$ . Thus  $\lambda(y) \in R$ ,  $b \in B$ , both  $\lambda(y)$  and  $b$  see  $x$  and  $(b, \lambda(y)) \in E_1$ . This completes the proof of Lemma 2

### 2.4 Relaxing the Disjointness Assumption

For ease of exposition, we have so far assumed that the set of possible guard locations has an empty intersection with the set of points to be guarded. Under this assumption, we have described a PTAS for the discrete terrain guarding problem. We now relax the assumption. The algorithm remains unchanged, and we indicate how the analysis is modified. For each  $x \in X$ , the point  $\lambda(x)$  (resp.  $\rho(x)$ ) denotes the leftmost (resp. rightmost) point that sees

$x$  among points in  $R \cup B$  *strictly* to the left (resp. right) of  $x$ , assuming such a point does exist. With this understanding, the construction of the planar graph  $\mathcal{G}$  proceeds with the sets  $A_1$ ,  $E_1$ ,  $A_2$ , and  $E_2$  defined exactly as above. One subtlety worth pointing out here is that when  $v \in R \cup B$  also belongs to  $X$ ,  $(\lambda(v), v)$  (resp.  $(v, \rho(v))$ ) is not considered to be above  $v$ .

There is a change in the construction of  $E_3$ . For each  $x \in X$  that is not in  $R \cup B$ , we proceed as before and add the edge  $(\lambda(x), \rho(x))$  to  $E_3$  if  $\lambda(x)$  and  $\rho(x)$  are of opposite colors. For  $x \in X$  that is also in  $R \cup B$ , we add the edge  $(\lambda(x), x)$  to  $E_3$  if  $\lambda(x)$  exists and  $\lambda(x)$  and  $x$  are of opposite colors; we also add the edge  $(x, \rho(x))$  to  $E_3$  if  $\rho(x)$  exists and  $\rho(x)$  and  $x$  are of opposite colors.

This completes the construction of the graph, which is readily seen to be planar. To show Lemma 2, we need to argue that for each  $x \in X$ , there are points  $r \in R$  and  $b \in B$  that see  $x$ , and  $(r, b) \in E_1 \cup E_2 \cup E_3$ . If  $x \notin R \cup B$  then the argument is exactly as before. Without loss of generality, assume that  $x \in R$ . Since  $B$  sees  $X$ , there must be a point in  $B$  that sees  $x$ . Assume that such a blue point lies to the left of  $x$ . The other case is symmetric. In this case, it must be that  $\lambda(x)$  exists. If  $\lambda(x) \in B$ , then we are done since we added  $(\lambda(x), x)$  to  $E_3$ . Therefore assume that  $\lambda(x) \in R$ . Let  $b$  be the leftmost blue point that sees  $x$ . Now we can use the reasoning in the last paragraph of the previous section to show that there is an edge  $(b, u) \in E_1$  such that both  $b$  and  $u$  see  $x$ ,  $b$  is blue, and  $u$  is red.

**Theorem 4** *The discrete terrain guarding problem admits a polynomial time approximation scheme.*

### 3 Conclusions

We have shown that the discrete terrain guarding problem admits a PTAS. We can also obtain a PTAS in the scenario where the possible guard locations are from a finite set, and we want to see the entire terrain – this problem is readily reduced to discrete terrain guarding. In the continuous terrain guarding problem, guards are allowed to be located anywhere on the terrain, and we wish to see the entire terrain. Here our local search algorithm returns a solution of cardinality at most  $(1 + \epsilon)$  times the optimal. However, it is not clear that the running time remains polynomial.

To explain the issue, consider a single update step of the local search. We start with a subset  $Q$  that sees the entire terrain. We want to determine if there exists subset  $S \subseteq Q$  of size at most  $b$  and subset  $S'$  of size at most  $|S| - 1$  such that  $(Q \setminus S) \cup S'$  guards the terrain. We can afford to enumerate all choices of  $S$ , so fix an  $S$  and consider the problem of determining if there is a corresponding  $S'$ . We want to know if at most  $|S| - 1$  guards can see the portion of the terrain not guarded by  $Q \setminus S$ . Even though  $|S|$  is a constant, it is not apparent that this task is polynomial-time solvable, because there are infinitely many guard locations. It turns out, however, that the task can be accomplished in polynomial-time, by an approach similar to Section 4 of [7].

Thus a single update step of the local search can be accomplished in polynomial time. However, the maximum number of bits needed to represent a guard in the updated

set (in particular  $S'$ ) is potentially larger than the maximum number of bits needed to represent a guard in  $Q$  before the update. For this reason, it is not clear that the running time of the sequence of update steps performed by the local search remains polynomial.

Turning now to 2.5D terrains, the local search algorithm like the one we use here will not yield a PTAS for the discrete case. An explicit counterexample is not too hard to construct.

We exploit the order claim in many of our arguments. Such an ordering is unavailable for 2.5D terrains. Moreover, as shown in [8], the 2.5D version does not admit a polynomial time algorithm with better than a logarithmic approximation factor, unless  $P = NP$ .

A very recent paper by Aschner et al. [1] notes that the analysis of [18] and [3] works not just for planar graphs but for graphs with small separators. The authors then present several applications of this. Among them is a PTAS for vertex guarding a polygon under an appropriate shallowness assumption.

## References

- [1] Rom Aschner, Matthew J. Katz, Gila Morgenstern, and Yelena Yuditsky. Approximation schemes for covering and packing. In *Proceedings of the Workshop on Algorithms and Computation, WALCOM' 13*, volume 7748 of *Lecture Notes in Computer Science*, pages 89–100, 2013.
- [2] Boaz Ben-Moshe, Matthew J. Katz, and Joseph S. B. Mitchell. A constant-factor approximation algorithm for optimal 1.5d terrain guarding. *SIAM J. Comput.*, 36(6):1631–1647, 2007.
- [3] Timothy M. Chan and Sariel Har-Peled. Approximation algorithms for maximum independent set of pseudo-disks. In *Proceedings of the 25th annual symposium on Computational geometry, SoCG '09*, pages 333–340, 2009.
- [4] Danny Z. Chen, Vladimir Estivill-Castro, and Jorge Urrutia. Optimal guarding of polygons and monotone chains. In *Proceedings of the 7th annual Canadian conference on computational geometry*, pages 133–138, 1995.
- [5] Kenneth L. Clarkson and Kasturi Varadarajan. Improved approximation algorithms for geometric set cover. In *Proceedings of the twenty-first annual symposium on Computational geometry, SoCG' 09*, pages 135–141, 2005.
- [6] Ajay Deshpande, Taejung Kim, Erik D. Demaine, and Sanjay E. Sarma. A pseudopolynomial time  $O(\log^2 n)$ -approximation algorithm for art gallery problems. In *Proceedings of the 10th Workshop on Algorithms and Data Structures, WADS' 07*, volume 4619 of *Lecture Notes in Computer Science*, pages 163–174, 2007.
- [7] Alon Efrat and Sariel Har-Peled. Guarding galleries and terrains. *Information Processing Letters*, 100(6):238 – 245, 2006.

- [8] Stephan Eidenbenz, Christoph Stamm, and Peter Widmayer. Inapproximability results for guarding polygons and terrains. *Algorithmica*, 31(1):79–113, 2001.
- [9] Khaled M. Elbassioni, Erik Krohn, Domagoj Matijevec, Julián Mestre, and Domagoj Severdija. Improved approximations for guarding 1.5-dimensional terrains. *Algorithmica*, 60(2):451–463, 2011.
- [10] Thomas Erlebach, Klaus Jansen, and Eike Seidel. Polynomial-time approximation schemes for geometric intersection graphs. *SIAM Journal on Computing*, 34(6):1302–1323, 2005.
- [11] Greg N. Frederickson. Fast algorithms for shortest paths in planar graphs, with applications. *SIAM J. Comput.*, 16(6):1004–1022, 1987.
- [12] Subir Ghosh. Approximation algorithms for art gallery problems. In *Proc. Canadian Information Processing Society Congress*, pages 429–434, 1987.
- [13] James King. A 4-approximation algorithm for guarding 1.5-dimensional terrains. In *Proc. of the 7th Latin American Symposium on Theoretical Informatics*, LATIN’ 06, pages 629–640, 2006.
- [14] James King and David G. Kirkpatrick. Improved approximation for guarding simple galleries from the perimeter. *Discrete & Computational Geometry*, 46(2):252–269, 2011.
- [15] James King and Erik Krohn. Terrain guarding is NP-hard. *SIAM J. Comput.*, 40(5):1316–1339, 2011.
- [16] Erik Krohn and Bengt J. Nilsson. Approximate guarding of monotone and rectilinear polygons. *Algorithmica*, 66(3):564–594, 2013.
- [17] D. Lee and A. Lin. Computational complexity of art gallery problems. *Information Theory, IEEE Transactions on*, 32(2):276–282, 1986.
- [18] Nabil H. Mustafa and Saurabh Ray. Improved results on geometric hitting set problems. *Discrete & Computational Geometry*, 44(4):883–895, 2010.