Introduction to Differential Forms in Tensor Calculus

Vicente Lira

University of Texas at San Antonio

Abstract

The purpose of this paper is to introduce differential forms in the study of tensor calculus. The reader should have general knowledge of vector calculus along with knowledge in advanced calculus and some linear algebra. The motivation behind this paper is to show students important introductory level concepts in differential forms along with a few concepts in tensor calculus by applying some important concepts in vector calculus.

1 Introduction

Assume we just finished a multivariable basis vector calculus class. Differential forms are not easy to understand right away. More so, it might not be immediately clear how differential forms are related or applied in calculus. For this reason, this paper focuses on explaining differential forms from a tensor calculus perspective. Albert Einstein used tensor calculus to formulate general relativity . Those not familiar with tensors or tensor calculus should not worry since this paper is an introduction to differential forms. Like all mathematics, tensor calculus takes much practice to grasp as a subject. For our purpose, we are treating differential forms as a new mathematical topic to the reader.

2 Tensors

We begin with a series of definitions to help the reader follow along with the process of solving the tensor problems. The idea of a tensor is fairly simple one. Students may initially think of a tensor as a generalization of a linear transformation to copies of a vector space.

Definition 2.1. Let V be a vector space, and let $V^N = V \times V \times ... \times V$. With $v_1, v_2, ..., v_{i+1}, v_{i-1}, ...v_n$ constant, let $f(v_1, v_2, ..., v, v_{i-1}, ..., v_n)$ be linear. In this case, f is said to be linear in the i-th variable. If f is linear in the i-th variable for $1 \leq i \leq n$, then we say that f is a *multilinear* function.

Definition 2.2. Let $\varphi : V^k \to \mathbb{R}$ be a function. We define φ to be a k-tensor on V if φ is multilinear.

Strictly speaking, we have defined a covariant tensor. This is the concept we need in order to discuss differential forms. For a deeper discussion of contravariant and covariant tensor see [2].

Definition 2.3. The set of all k-tensors on a vector space V is denoted $T^k(V)$. For φ, $\eta \in T^k(V)$, and $c \in \mathbb{R}$, we define

$$\begin{aligned}(\varphi + \eta)(v_1, ..., v_k) &= \varphi(v_1, ..., v_k) + \eta(v_1, ..., v_k)\\ (c\varphi)(v_1, ..., v_k) &= c(\varphi(v_1, ..., v_k))\end{aligned}$$

With the definitions stated above we can say that for $k \in \mathbb{N}$, $T^k(V)$ is in fact a vector space. We can use the standard scalar multiplication and addition to see that we can satisfy the axioms of a vector space. The function whose value is zero on every k-tuple of our vector space is known as the zero element of V. Based on our definitions above, $T^1(V)$ is the set of all linear transformations $T : V \to \mathbb{R}$, and we set $T^0(V) = \mathbb{R}$. The following lemma tells us that tensors are uniquely determined by their values on basis elements.

Lemma 2.1. Let $b_1, ..., b_n$ be a basis for a vector space V. Let $\varphi, \eta : V^K \to \mathbb{R}$ be k-tensors on V satisfying $\varphi(b_{i_1}, ..., b_{i_k}) = \eta(b_{i_1}, ..., b_{i_k})$ for every k-tuple $I = (i_1, ...i_k)$, where $1 \leq i_m \leq n$. Then $\varphi = \eta$.

The proof for this lemma is trivial.

Differential forms are essentially alternating (i.e. completely antisymmetric) tensors. For example, a 2-tensor $\varphi \in T^2(V)$ is alternating if

$$\varphi(u, v) = -\varphi(v, u)$$

for all $u, v \in V$. Now lets consider a 3-tensor $\varphi \in T^3(V)$. What could it mean to say that φ is completely antisymmetric? Based on Definition 2.3 and Lemma 2.1, it makes sense to require the relationships

$$\begin{aligned} \varphi(u, v, w) &= -\varphi(v, u, w) \\ \varphi(u, v, w) &= -\varphi(u, w, v) \\ \varphi(u, v, w) &= -\varphi(w, v, u) \end{aligned} \tag{1}$$

i.e. the interchange of any two arguments of the 3-form φ introduces a minus sign. To help the reader see the usefulness of (1), we prove the cyclic permutation rule

$$\varphi(u, v, w) = \varphi(v, w, u) = \varphi(w, u, v).$$

To prove $\varphi(u, v, w) = \varphi(v, w, u)$, we note that $\varphi(u, v, w) = -\varphi(v, u, w) = \varphi(v, w, u)$ by applying (1) twice.

Definition 2.4. A permutation σ of a set A is a bijection from A to itself. The set of all permutations of $\{1, ..., k\}$ is denoted by S_k.

An easy way to think about permutations is in terms of order. The operation of a permutation essentially changes the structure of the order of the elements of the set. In mathematics, particularly in differential geometry, linear algebra, and tensor calculus there is a symbol known as the *Levi-Civita* symbol, named after the Italian mathematician and physicist Tullio Levi-Civita , and it represents a function that maps ordered n-tuples of integers in $\{1, 2, ..., n\}$ to the set $\{1, -1, 0\}$. These numbers are defined from the sign of a permutation of natural numbers $1, 2, ..., n$, for some positive integer n. The standard notation of Levi-Civita symbol is the Greek lower case epsilon ε or ϵ, or, less commonly, the Latin lower case e. Its value is defined by

$$\epsilon_{i_1 i_2 \ldots i_n} = \begin{cases} +1 & (i_1, \ldots, i_n) \text{is an even permutation of } (1, 2, \ldots, n) \\ -1 & (i_1, \ldots, i_n) \text{is an even permutation of } (1, 2, \ldots, n) \\ 0 & \text{if } i_p = i_q \text{ for some } p \neq q \end{cases}.$$

Index notation allows one to display permutations in a way compatible with tensor analysis. For example, the Levi-Civita symbol in two dimensions $n = 2$ has the properties

$$\varepsilon_{12} = 1, \varepsilon_{21} = -1, \varepsilon_{11} = 0, \varepsilon_{22} = 0.$$

This rule is necessary because we want an alternating multilinear form to behave so that switching any two arguments imposes a negative sign. Applying an even permutation to the coordinates is equivalent to applying an even number of switches, hence applying an even number of negative signs, which does nothing. With these preliminaries, we can state the definition of a k-form:

Definition 2.5. Let $\varphi \in T^k(V)$. φ is called a k-form, or alternating multilinear form if

$$\varphi(v_1, \ldots, v_k) = \varepsilon_{\sigma(1)\ldots\sigma(k)}\varphi(v_{\sigma(1)}, \ldots, v_{\sigma(k)})$$

for all permutations σ of $1, \ldots, n$.

For the following definition 2.6, let M be an open set in ordinary n-dimensional Euclidean space.

Definition 2.6. A mapping ω that associates to each $x \in M$ an alternating k-tensor $\omega(x)$ and such that $x \to \omega$ is smooth is called a $k-$form. The space of k-forms on M is denoted $\Omega^k(M)$.

The definition generalizes to manifolds: a k-form ω on a differentiable manifold M is a smooth section of the bundle of alternating k-tensors on M. For a more thorough discussion of manifolds, see [1].

For our purpose, we will say a 1-form is a *covector* field and a 0-form as being a smooth function on M, so $\Omega^0(M) = C^\infty(M)$ (= infinitely differentiable functions on M). Next, we consider an example in three-dimensional Euclidean space. A differential 1-form in the three-dimensional space is an expression

$$\omega = F(x, y, z)dx + G(x, y, z)dy + H(x, y, z)dz \tag{2}$$

where F, G, and H are functions on an open set M. How does this tie in with our definition of forms? To see the connection, we take $V = \mathbb{R}^3$ and define dx, dy, dz to be forms that act as follows:

$$\begin{aligned} dx(\boldsymbol{i}) = 1,\ dx(\boldsymbol{j}) = 0, dx(\boldsymbol{k}) = 0 \\ dy(\boldsymbol{i}) = 0,\ dx(\boldsymbol{j}) = 1, dy(\boldsymbol{k}) = 0 \\ dz(\boldsymbol{i}) = 0,\ dz(\boldsymbol{j}) = 0, dz(\boldsymbol{k}) = 1 \end{aligned}$$

where $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ are the three standard basis vectors of $\mathbb{R}^3$. So ω acts on a vector $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$ in the following way:

$$\begin{aligned} \omega(\boldsymbol{a}) &= F(x, y, z)dx(\boldsymbol{a}) + G(x, y, z)dy(\boldsymbol{a}) + H(x, y, z)dz(\boldsymbol{a}) \\ &= F(x, y, z)a_1 + G(x, y, z)a_2 + H(x, y, z)a_3. \end{aligned}$$

This process is useful to discuss the differential of a function $f(x, y, z)$ defined on M, which is defined as

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz. \tag{3}$$

To see that (3) makes sense and has a familiar interpretation, we let df act on a unit vector $\boldsymbol{a}$:

$$df(\boldsymbol{a}) = \frac{\partial f}{\partial x}a_1 + \frac{\partial f}{\partial y}a_2 + \frac{\partial f}{\partial z}a_3 = \nabla f(x, y, z) \cdot \boldsymbol{a},$$

i.e. $df(\boldsymbol{a})$ gives the directional derivative of f in direction $\boldsymbol{a}$. The meaning of 2-forms in three-dimensional Euclidean space will be discussed further in 4 in connection with the concept of integration.

Next, we discuss an important product operation for forms, called the wedge product. The wedge product is an operator which takes a k-form and an j-form to a $k+j$-form, that is associative, distributive and anticommutative. It is uniquely determined by the properties that follow - for a constructive definition of the wedge product, we refer to [1].

An important consequence of antisymmetry is that the wedge of any 1-form with itself is zero:

$$\alpha \wedge \alpha = -\alpha \wedge \alpha = 0$$

However, it is imperative to know that the previous statement is not purely an algebraic fact. The reason the wedge of two 1-forms is zero is that it represents projection onto a plane of zero area. Assuming the wedge product is associative and distributive, we can always wedge together any two forms. The wedge product of a p-form with a q-form is a (p+q)-form.

Definition 2.7. We will use the symbol $\wedge$, known as the wedge, as a binary operation on differential forms called the *wedge product*. The wedge product has the following properties for any k-form α, l-form β, and m-form γ:

- **Antisymmetry**: $\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha$
- **Associativity**: $\alpha \wedge (\beta \wedge \gamma) = (\alpha \wedge \beta) \wedge \gamma$
- **Homogeneity**: $(c\alpha) \wedge \beta = c(\alpha \wedge \beta)$ for any real number c

And in the case where $l = m$, we have

- **Distributivity**: $\alpha \wedge (\beta + \gamma) = \alpha \wedge \beta + \alpha \wedge \gamma$

Here is an example:
We calculate the wedge product of two 1-forms ω, η in $\mathbb{R}^2$

$$\omega = F dx + G dy, \quad \eta = U dx + V dy.$$

We write

$$\begin{aligned} \omega \wedge \eta &= (F dx + G dy) \wedge (U dx + V dy) \\ &= FU dx \wedge dx + FV dx \wedge dy + GU dy \wedge dx + GV dy \wedge dy \end{aligned}$$

Since $dx \wedge dx = dy \wedge dy = 0$ and $dx \wedge dy = -dy \wedge dx$,

$$\omega \wedge \eta = (FV - GU) dx \wedge dy. \tag{4}$$

The wedge product is associative, but not generally commutative. The wedge product is uniquely characterized by the properties of associativity, distributivity, homogeneity and anticommutativity (see [1]).

Finally, we discuss briefly the concept of exterior derivative, where for simplicity, we work with differential forms in $\mathbb{R}^3$. Every 1-form in $\mathbb{R}^3$ can be written as $(dx^1 = dx, dx^2 = dy, dx^3 = dz)$

$$\omega = \omega_1 dx^1 + \omega_2 dx^2 = \sum_{i=1}^{2} \omega_i dx^i.$$

Every 2-form X can be written as

$$\eta = \eta_{12} dx^1 \wedge dx^2 + \eta_{23} dx^2 \wedge dx^3 + \eta_{13} dx^1 \wedge dx^3 = \sum_{i<j} \eta_{ij} dx^i \wedge dx^j$$

and more generally, a k-form ζ is

$$\zeta = \sum_{i_1<i_2<\ldots<i_k} \zeta_{i_1 i_2 \ldots i_k} dx^{i_1} \wedge dx^{i_2} \wedge \ldots dx^{i_k}.$$

Definition 2.8. The exterior derivative of a differential form of degree k is a differential form of degree $k+1$. If f is a smooth function (a 0-form), then the exterior derivative of f is the differential of f. If ω is a k-form, we write

$$\omega = \sum_{i_1<i_2<\ldots<i_k} \omega_{i_1 i_2 \ldots i_k} dx^{i_1} \wedge dx^{i_2} \wedge \ldots dx^{i_k}.$$

and define the exterior derivative $d\omega$ by the formula

$$d\omega = \sum_{i_1<i_2<\ldots<i_k} d\omega_{i_1 i_2 \ldots i_k} \wedge dx^{i_1} \wedge dx^{i_2} \wedge \ldots dx^{i_k}. \tag{5}$$

We should note the appearance of an additional $\wedge$ between the differential of the coefficients and the remaining dx^i's.

Here is an example:
Suppose we are given the 1-form

$$\omega = x^2 dx + xy dy + dz$$

and we need to compute $d\omega$. Definition 2.8 above tells us that we need the differentials of the coefficients x^2, xz and 1. So

$$d(x^2) = 2x dx, \quad d(xy) = y dx + x dy, \quad d(1) = 0.$$

Hence

$$\begin{aligned} d\omega &= 2x dx \wedge dx + (y dx + x dy) \wedge dy + d(1) \wedge dz \\ &= y dx \wedge dy \end{aligned}$$

where we have used $dx \wedge dx = 0, dy \wedge dy = 0$.

3 Differential forms and signed areas/volumes

In this section, we discuss a connection between the wedge product and ordinary vectors. Up to now, we defined the wedge product in relation to differential forms. We can, however, associate any vector $\boldsymbol{u}$ with a form, if a symmetric inner product is given. For $\mathbb{R}^3$, we can take the inner product to be the usual dot product between vectors: $\boldsymbol{u} \cdot \boldsymbol{v}$.

Definition 3.1. For any $\boldsymbol{u} \in \mathbb{R}^3$, we define a 1-form $\omega_{\boldsymbol{u}}$ by

$$\omega_{\boldsymbol{u}}(\boldsymbol{v}) = \boldsymbol{u} \cdot \boldsymbol{v}.$$

We should note that $\omega_{\boldsymbol{u}}$ is defined by the way it acts on $\boldsymbol{v}$, and yields a scalar. It is easy to check that if u_1, u_2, u_3 are the components of $\boldsymbol{u}$, $\omega_{\boldsymbol{u}}$ is

$$\omega = u_1 dx + u_2 dy + u_3 dz.$$

As an example, we can then take the wedge product of $\omega_{\boldsymbol{a}}, \omega_{\boldsymbol{b}}$ to get

$$\begin{aligned}\omega_{\boldsymbol{a}} \wedge \omega_{\boldsymbol{b}} &= (a_1 dx + a_2 dy + a_3 dz) \wedge (b_1 dx + b_2 dy + b_3 dz)\\ &= (a_2 b_3 - a_3 b_2) dy \wedge dz + (a_3 b_1 - a_1 b_3) dz \wedge dx + (a_1 b_2 - a_2 b_1) dx \wedge dy\end{aligned}$$

Note that the wedge product $\omega_{\boldsymbol{u}} \wedge \omega_{\boldsymbol{v}}$ contains the same information as the cross product $\boldsymbol{u} \times \boldsymbol{v}$: it simply equals $\omega_{\boldsymbol{u}\times\boldsymbol{v}}$. It is interesting to see what happens when we compute the wedge product of three one-forms $\omega_{\boldsymbol{u}}, \omega_{\boldsymbol{v}}, \omega_{\boldsymbol{w}}$:

$$\omega_{\boldsymbol{u}} \wedge \omega_{\boldsymbol{v}} \wedge \omega_{\boldsymbol{w}} = \det[\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}] dx \wedge dy \wedge dz. \tag{6}$$

So the wedge product can also be used to compute determinants. It should be noted that the algebraic properties of the wedge product are consistent with the way signed volumes and determinants behave. For example, the signed volume of the parallelepiped defined by three vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ in $\mathbb{R}^3$ is given by their determinant - if we reverse the order of two vectors $\boldsymbol{u} \leftrightarrow \boldsymbol{v}$ for example, the determinant changes its sign. Moreover, if say $\boldsymbol{u} = \boldsymbol{v}$, we get

$$\omega_{\boldsymbol{u}} \wedge \omega_{\boldsymbol{v}} \wedge \omega_{\boldsymbol{w}} = 0$$

because $\omega \wedge \omega = 0$ for any 1-form ω. This form reflects the fact that the volume of the corresponding parallelopided becomes zero.

There is also a related concept of bi- and multivectors (see [3]). These work by directly defining wedge products of vectors, instead of forms.

4 Differential forms and surfaces

In this section, we see an important application of differential forms: They can be integrated over manifolds. A manifold is a generalization of a surface. More specifically, an n-dimensional manifold is a set that looks like $\mathbb{R}^n$. It is a union of subsets each of which may be equipped with a coordinate system with coordinates running over an open subset on $\mathbb{R}^n$. Consider a surface S in $\mathbb{R}^3$ parameterized by a vector function

$$\boldsymbol{r}(u, w)$$

where u, w are in $[0, 1] \times [0, 1]$. Consider now a 2-form

$$\omega = a(x, y, z) dx \wedge dy + b(x, y, z) dy \wedge dz + c(x, y, z) dx \wedge dz$$

in $\mathbb{R}^3$, with variable coefficients a, b, c. Given a point (x, y, z) and two vectors $\boldsymbol{k}, \boldsymbol{l}$, we can imagine a small parallelogram spanned by the vectors $\boldsymbol{k}\Delta u, \boldsymbol{l}\Delta w$ attached at the point (x, y, z). Here, $\Delta u, \Delta w$ are small positive numbers. Evaluating

$$\omega(\boldsymbol{k}\Delta u, \boldsymbol{l}\Delta w) = \omega(\boldsymbol{k}, \boldsymbol{l}) \Delta u \Delta w \tag{7}$$

gives a real number. We may imagine the 2-form ω describing the flux density of a physical quantity, for example the mass flow of a fluid. In that case, (7) gives the mass of fluid crossing the small

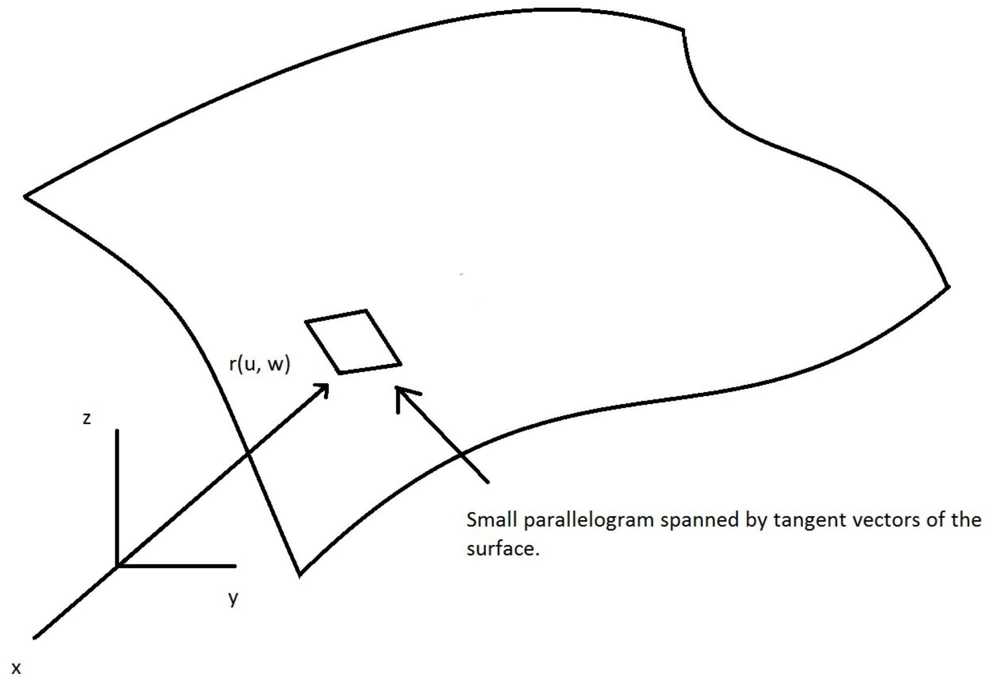

Figure 1: A surface in three-dimensional space

parallelogram per unit time. Close to any point of the surface S, the surface can be approximated by a small parallelogram as in Figure 1, where $\boldsymbol{k}, \boldsymbol{l}$ are the surface tangent vectors $\frac{\partial \boldsymbol{r}}{\partial u}, \frac{\partial \boldsymbol{r}}{\partial w}$. The total flux through the surface should then be the sum of terms of the form (7). The Riemann integral, this suggests defining

$$\int_S \omega := \iint_{[0,1]\times[0,1]} \omega \left(\frac{\partial \boldsymbol{r}}{\partial u}, \frac{\partial \boldsymbol{r}}{\partial w} \right) \, dudw \tag{8}$$

as the flux through the surface. So we see that a 2-form may be integrated over a 2-dimensional surface, and more generally an n-form can be integrated over an n-dimensional manifold. A development of these concepts leads directly to Stokes' theorem [1].

5 Examples of Differential Forms

In this section, we show some concrete computations and examples using differential forms.

Example 1. Let $\boldsymbol{a}$ be a vector in $\mathbb{R}^3$. Consider a two-form φ given by:

$$\varphi(\boldsymbol{v}, \boldsymbol{w}) = \det(\boldsymbol{a}, \boldsymbol{v}, \boldsymbol{w}) \quad (\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^3)$$

We write this two-form as a linear combination of fundamental forms, expressed as coordinates of $\boldsymbol{a}$.

Solution.
If we write

$$\boldsymbol{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \quad \boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \quad \boldsymbol{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}.$$

Then we write

$$\begin{aligned} \varphi(\boldsymbol{v}, \boldsymbol{w}) &= \begin{vmatrix} a_1 & v_1 & w_1 \\ a_2 & v_2 & w_2 \\ a_3 & v_3 & w_3 \end{vmatrix} \\ &= a_1 \begin{vmatrix} v_2 & w_2 \\ v_3 & w_3 \end{vmatrix} - a_2 \begin{vmatrix} v_1 & w_1 \\ v_3 & w_3 \end{vmatrix} + a_3 \begin{vmatrix} v_1 & w_1 \\ v_2 & w_2 \end{vmatrix} \end{aligned}$$

We can see that (see (4))

$$\begin{aligned} dx \wedge dy(\boldsymbol{v}, \boldsymbol{w}) &= \det \begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix} \\ dy \wedge dz(\boldsymbol{v}, \boldsymbol{w}) &= \det \begin{bmatrix} v_2 & w_2 \\ v_3 & w_3 \end{bmatrix} \\ dx \wedge dz(\boldsymbol{v}, \boldsymbol{w}) &= \det \begin{bmatrix} v_1 & w_1 \\ v_3 & w_3 \end{bmatrix} \end{aligned}$$

By substitution, we have

$$\varphi(\boldsymbol{v}, \boldsymbol{w}) = a_1 dy \wedge dz(\boldsymbol{v}, \boldsymbol{w}) - a_2 dx \wedge dz(\boldsymbol{v}, \boldsymbol{w}) + a_3 dx \wedge dy(\boldsymbol{v}, \boldsymbol{w})$$

Therefore,

$$\varphi = a_1 dy \wedge dz - a_2 dx \wedge dz + a_3 dx \wedge dy$$

Example 2. This example involves integration on manifolds and Stokes' Theorem (see [1]). Let U be a compact manifold of dimension 3 in $\mathbb{R}^3$ with boundary U and a two-finite volume. The volume is given by

$$\mathrm{Vol}_3\, U = \frac{1}{3} \int_{\partial U} x_3 dx_1 \wedge dx_2 + x_2 dx_3 \wedge dx_1 + x_1 dx_2 \wedge dx_3$$

Solution. Let $w = \dfrac{1}{3}(x_3 dx_1 \wedge dx_2 + x_2 dx_3 \wedge dx_1 + x_1 dx_2 \wedge dx_3)$. We can compute the exterior derivative term by term. For example,

$$\begin{aligned} d(x_3 dx_1 \wedge dx_2) &= \left(\frac{\partial}{\partial x_1} x_3 + \frac{\partial}{\partial x_2} x_3 + \frac{\partial}{\partial x_3} x_3 \right) dx_1 \wedge dx_2 \\ &= (0 + 0 + 1 dx_3) \wedge dx_1 \wedge dx_2. \end{aligned}$$

So after a calculation, we can say

$$dw = \frac{1}{3}\left(dx_3 \wedge dx_1 \wedge dx_2 + dx_2 \wedge dx_3 \wedge dx_1 + dx_1 \wedge dx_2 \wedge dx_3\right)$$

Using the property $dy \wedge dx = -dx \wedge dy$, we have:

$$\begin{aligned} dw &= \frac{1}{3}\Big(dx_1 \wedge dx_2 \wedge x_3 + dx_1 \wedge dx_2 \wedge dx_3 + dx_1 \wedge dx_2 \wedge dx_3\Big) \\ &= \frac{1}{3}\Big(3dx_1 \wedge dx_2 \wedge dx_3\Big) \\ &= dx_1 \wedge dx_2 \wedge dx_3 \end{aligned}$$

So we can say by Stokes' Theorem that

$$\int_{\partial U} \frac{1}{3}\left(x_3 dx_1 \wedge dx_2 + x_2 dx_3 \wedge dx_1 + x_1 dx_2 \wedge dx_3\right) = \int_U dx_1 \wedge dx_2 \wedge dx_3.$$

$dx_1 \wedge dx_2 \wedge dx_3$ is the form that defines the volume in Euclidean space (see e.g.[2]).

Example 3. Find a 1-form φ such that $d\varphi = ydz \wedge dx - xdy \wedge dz$. Write $\varphi = adx + bdy + cdz$ (1-form) where a, b, c are functions of x, y, z.

Solution.

$$\begin{aligned} d\varphi &= \Big(\frac{\partial a}{\partial y}dy + \frac{\partial a}{\partial z}dz\Big) \wedge dx + \Big(\frac{\partial b}{\partial x}dx + \frac{\partial b}{\partial z}dz\Big) \wedge dy + \Big(\frac{\partial c}{\partial x}dx + \frac{\partial c}{\partial y}dy\Big) \wedge dz + \frac{\partial c}{\partial y}dy \wedge dz \\ &= \Big(\frac{\partial b}{\partial x} - \frac{\partial a}{\partial y}\Big)dx \wedge dy + \Big(\frac{\partial c}{\partial x} - \frac{\partial a}{\partial z}\Big)dx \wedge dz + \Big(\frac{\partial c}{\partial y} - \frac{\partial b}{\partial z}\Big)dy \wedge dz \\ &= 0 + (-y)dx \wedge dz + (-x)dy \wedge dz \\ &= ydz \wedge dx = xdy \wedge dz \end{aligned}$$

For $a = xy$, $b = xz$, and $c = 0$:

$$\begin{aligned} \Big(\frac{\partial b}{\partial x} - \frac{\partial a}{\partial y}\Big) &= 0 \Rightarrow \frac{\partial b}{\partial x} = \frac{\partial a}{\partial y} \\ \Big(\frac{\partial c}{\partial x} - \frac{\partial a}{\partial z}\Big) &= -y \Rightarrow \frac{\partial c}{\partial x} - \frac{\partial b}{\partial y} = -y \\ \Big(\frac{\partial c}{\partial y} - \frac{\partial b}{\partial z}\Big) &= -x \Rightarrow \frac{\partial c}{\partial y} - \frac{\partial a}{\partial z} = -x \end{aligned}$$

Therefore,

$$\varphi = yzdx + xzdy.$$

6 Conclusion

In this paper, we discussed the definition of differential forms along with some examples. For further reading, we recommend [1, 2].

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