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Multiterm Impulsive Caputo–Hadamard Type Differential Equations of Fractional Variable Order

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Abstract: In this study, we deal with an impulsive boundary value problem (BVP) for differential equations of variable fractional order involving the Caputo–Hadamard fractional derivative. The fundamental problems of existence and uniqueness of solutions are studied, and new existence and uniqueness results are established in the form of two fixed point theorems. In addition, Ulam–Hyers stability sufficient conditions are proved illustrating the suitability of the derived fundamental results. The obtained results are supported also by an example. Finally, the conclusion notes are highlighted.

Keywords: Caputo–Hadamard fractional derivative; variable order; impulses; existence of solutions; uniqueness; fixed point theorem; Ulam–Hyers stability

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1. Introduction

The idea of fractional-order integration and differentiation goes back to sixteenth century [1,2]. Since then, the attention to fractional calculus greatly increased in relation to modeling and control of numerous real processes using functions with fractional derivatives. Such a generalization of the notion allows the use of the important features of fractional derivatives, such as more degrees of freedom and infinite memory. In fact, fractional-order systems are characterized by infinite memory, as opposed to integer-order systems. Additionally, fractional integrals can be used to describe the fractal media [3].

In recent years, fractional differential equations have been actively used as models of numerous real-world phenomena studied in science, biology, engineering, and economics. In fact, fractional-order differential equations are widely applied in material and quantum mechanics, signal processing and systems identification, anomalous diffusion, wave propagation, etc. [4,5]. The efficient use of these equations in mathematical modeling requires the development of their fundamental and qualitative theories [6–8]. The progress in this development is related to the investigation of various type of fractional derivatives, such as Riemann–Liouville, Caputo, Hadamard, Grunwald–Letnikov, Marchaud, and Riesz just to name a few [2]. The books [6,8] provide an excellent summary on the subject.

Recently, the topic of fractional equations has been expanded and different new classes of equations have been introduced. One of the most extensively studied classes of fractional differential equations is the class of fractional equations with variable fractional order [9,10]. Different researchers studied properties of fractional equations with variable fractional order of Riemann–Liouville type [11–13], Caputo type [14–16], and Hadamard type [17–19]. The enormous interest in these equations is due mainly to the extended possibilities of their applications [20–22]. A very good overview of some main applications of variable-order fractional operators has been given in [23].

The applicability of the hybrid Caputo–Hadamard type fractional derivatives is the main reason for the research interest in their theory [24–26]. Very recently, a few authors have studied the properties of the extension of such derivatives to a variable order [27,28].

From the other side, the apparatus of impulsive differential equations has been widely used in the description of processes with abrupt changes during their evolution [29–31]. The theory of such equations is also very well applied in impulsive control problems [32].

Additionally, the theory of impulsive differential equations has been extended to the fractional-order case. Numerous impulsive fractional-order systems with constant fractional derivatives have been proposed and their dynamical properties have been studied [33–38].

In particular, Benchohra et al. studied in [39] the following problem

$$^{c}D^{\tau}_{M^{+}_{\vartheta}}x(t) = \Psi(t, x(t)), \text{ for each }, \ t \in [0, M], \ t \neq M_{\vartheta}, \ \vartheta = 1, \ldots, n,$$

$$\Delta x|_{t=M_{\vartheta}} = \Phi_{\vartheta}(x(M^{-}_{\vartheta})), \ \vartheta = 1, \ldots, n,$$
$$x(0) = x_0,$$

where Ψ, Φ_{ϑ} are given functions, $\Delta x = x(t^{+}) - x(t^{-})$, $^{c}D^{\tau}_{M^{+}_{\vartheta}}$ illustrates the Caputo fractional derivative of a constant order τ, $0 < \tau \leq 1$, given as

$$^{c}D^{\tau}_{\rho_1^{+}}\eta(t) = \frac{1}{\Gamma(1-\tau)}\int_{\rho_1}^{t}\frac{\eta'(\rho)}{(t-\rho)^{\tau}}d\rho, \ t > \rho_1$$

for a function η and Γ denotes the Gamma function.

Correspondingly, results on impulsive variable-order fractional differential equations are reported very seldom [40]. In addition, the existing results on impulsive fractional differential equations involving constant Caputo–Hadamard type derivatives are very few [41,42]. There are no results reported for impulsive fractional Caputo–Hadamard fractional differential equations with variable order fractional derivatives. The aim of the presented research is to introduce some fundamental results for such equations. We expect that our contribution will motivate more researchers to develop the theory.

Inspired by [11,14,17,19,37,39–41], we deal with the following impulsive boundary value problem (BVP)

$$^{c}D^{\tau(t)}_{M^{+}_{\vartheta}}x(t) = \Psi(t, x(t), I^{\tau(t)}_{M^{+}_{\vartheta}}x(t)), \text{ for } t \in \varpi := [1, M], \ M > e, \ t \neq M_{\vartheta}, \ \vartheta = 1, \ldots, n, \quad (1)$$

$$\Delta x|_{t=M_{\vartheta}} = \Phi_{\vartheta}(x(M^{-}_{\vartheta})), \ \vartheta = 1, \ldots, n, \quad (2)$$

$$ax(1) + bx(M) = c, \quad (3)$$

where $0 < \tau(t) \leq 1$, $\Psi : \varpi \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $\Phi_{\vartheta} : \mathbb{R} \to \mathbb{R}$, $\vartheta = 1, \ldots, n$ are continuous functions, $^{c}D^{\tau(t)}_{M^{+}_{\vartheta}}$ and $I^{\tau(t)}_{M^{+}_{\vartheta}}$ illustrate the Caputo–Hadamard derivative and the Hadamard integral operators of variable order $\tau(t)$, respectively, a, b, and c are real constants with $a + b \neq 0$, $1 = M_0 < M_1 < \cdots < M_n < M_{n+1} = M$, $\Delta x|_{t=M_{\vartheta}} = x(M^{+}_{\vartheta}) - x(M^{-}_{\vartheta})$, $x(M^{+}_{\vartheta}) = \lim_{h\to 0^{+}} x(M_{\vartheta} + h)$ and $x(M^{-}_{\vartheta}) = \lim_{h\to 0^{-}} x(M_{\vartheta} + h)$ represent the right-hand side and left-hand side limits of $x(t)$ at $t = M_{\vartheta}$, $\vartheta = 1, \ldots, n$, respectively.

The main contributions of our research are:

1. We introduce a BVP for a class of impulsive Caputo–Hadamard fractional differential equations with fractional derivatives of variable order;
2. Existence and uniqueness criteria for the introduced BVPs are established;
3. As an application, results on Ulam–Hyers stability of the solutions are proposed;
4. An example is developed to demonstrate our results.

The organization of the rest of this paper is as follows. Some definitions and auxiliary results are given in Section 2. In Section 3, the main existence and uniqueness results for solutions of the BVP (1)–(3) of impulsive fractional Caputo–Hadamard fractional differential equations with fractional derivatives of variable order are proposed. The criteria proposed are presented in the form of two fixed point theorems. Section 4 is devoted to our main Ulam–Hyers stability results. One example is presented in Section 5, to show the efficiency and validity of the proposed results. Finally, some conclusion notes are given in Section 6.

2. Auxiliary Results

In this section, we list some definitions and propositions that are used in the following sections.

For $k \in \mathbb{N}$ we denote by $AC^k(\varpi)$ the set

$$AC^k(\varpi) = \{x : \varpi \to \mathbb{R},\ x, x', \ldots, x^{(k-1)} \in C(\varpi, \mathbb{R}),\ x^{k-1} \text{ is absolutely continuous}\}$$

and for $0 < \rho_1 < \rho_2 < \infty$, $\delta = t\frac{d}{dt}$, we denote by $AC^k_\delta([\rho_1, \rho_2])$ the set

$$AC^k_\delta([\rho_1, \rho_2]) = \{x : [\rho_1, \rho_2] \to \mathbb{R},\ \delta^{(k-1)}x(t) \in AC([\rho_1, \rho_2])\}.$$

Consider the sets of functions

$PC(\varpi, \mathbb{R}) = \{x : \varpi \to \mathbb{R}, x \in C((M_\vartheta, M_{\vartheta+1}], \mathbb{R})$, there exist $x(M^-_\vartheta)$ and $x(M^+_\vartheta)$, $\vartheta = 1, \ldots, n$ with $x(M^-_\vartheta) = x(M_\vartheta)\}$;

$PC^1(\varpi, \mathbb{R}) = \{x \in PC(\varpi, \mathbb{R}),\ x \in C^1((M_\vartheta, M_{\vartheta+1}], \mathbb{R})$, there exist $x'(M^-_\vartheta)$ and $x'(M^+_\vartheta)$, $\vartheta = 1, \ldots, n$ with $x'(M^-_\vartheta) = x'(M_\vartheta)\}$.

Note that the set $PC(\varpi, \mathbb{R})$ is a Banach space with a norm defined as

$$\|x\| = \sup\{|x(t)| : t \in \varpi\}.$$

For $1 \le \rho_1 < \rho_2 < \infty$, we consider the mapping $\tau(t) : [\rho_1, \rho_2] \to (0, 1]$. The Hadamard fractional integral (HFI) of variable order $\tau(t)$ for $\eta(t) \in L^1(\varpi, \mathbb{R})$ [9,27,28] is

$$I^{\tau(t)}_{\rho_1^+}\eta(t) = \frac{1}{\Gamma(\tau(t))}\int_{\rho_1}^t (\log\frac{t}{\varrho})^{\tau(t)-1}\eta(\varrho)\frac{d\varrho}{\varrho},\ t > \rho_1, \tag{4}$$

and the Caputo-type Hadamard fractional derivative (CHFD) of variable order $\tau(t)$ for $\eta(t) \in AC^1([\rho_1, \rho_2])$ [28] is

$$\begin{aligned} {}^cD^{\tau(t)}_{\rho_1^+}\eta(t) = {} & \frac{t\tau'(t)}{\Gamma(2-\tau(t))}\int_{\rho_1}^t (\log\frac{t}{\varrho})^{1-\tau(t)}\eta'(\varrho)\Big[\frac{1}{1-\tau(t)} - \log(\log\frac{t}{\varrho})\Big]d\varrho \\ & + \frac{1}{\Gamma(1-\tau(t))}\int_{\rho_1}^t (\log\frac{t}{\varrho})^{-\tau(t)}\eta'(\varrho)d\varrho,\ t > \rho_1. \end{aligned} \tag{5}$$

It is clear that, if $\tau(t)$ is a constant function, $\tau(t) = \tau$, then HFI and CHFD are reduced to the classical Hadamard integral $I^\tau_{\rho_1^+}$ and Caputo-type Hadamard derivative ${}^cD^\tau_{\rho_1^+}$, respectively [9,27].

Next, we will present some important properties of ${}^cD^\tau_{\rho_1^+}$ and $I^\tau_{\rho_1^+}$.

Proposition 1 ([6]). *Let $\tau_1,\ \tau_2 > 0$, $\rho_1, \rho_2 > 0$, $\eta \in AC^k_\delta([\rho_1, \rho_2])$. Then,*

$$I^{\tau_1}_{\rho_1^+}\,{}^cD^{\tau_1}_{\rho_1^+}\eta(t) = \eta(t) - \sum_{\vartheta=0}^{k-1}\frac{\delta^\vartheta\eta(\varrho)}{\vartheta!}(\log\frac{t}{\varrho})^\vartheta$$

and

$$I^{\tau_1}_{\rho_1^+}I^{\tau_2}_{\rho_1^+}\eta(t) = I^{\tau_2}_{\rho_1^+}I^{\tau_1}_{\rho_1^+}\eta(t) = I^{\tau_1+\tau_2}_{\rho_1^+}\eta(t).$$

Remark 1 ([19]). *In the general case*

$$I_{\rho_1^+}^{\tau_1(t)} I_{\rho_1^+}^{\tau_2(t)} \eta(t) \neq I_{\rho_1^+}^{\tau_1(t)+\tau_2(t)} \eta(t).$$

Proposition 2 ([19]). *Let $\tau \in C(\varpi, (0,1])$ and $0 \leq \gamma \leq \min_{t\in\varpi} |\tau(t)|$. Then, for $\eta \in C_\gamma(\varpi, \mathbb{R})$ where*

$$C_\gamma(\varpi, \mathbb{R}) = \{\eta(t) \in C(\varpi, \mathbb{R}),\ (\log t)^\gamma \eta(t) \in C(\varpi, \mathbb{R})\},$$

the (HFI) $I_{0^+}^{\tau(t)} \eta(t)$ exists for any $t \in \varpi$.

Proposition 3 ([19]). *If $\tau \in C(\varpi, (0,1])$, then, $I_{0^+}^{\tau(t)} \eta(t) \in C(\varpi, \mathbb{R})$ for any $\eta \in C(\varpi, \mathbb{R})$.*

Proposition 4 ([15,43]). *Let $\tau \in [0,1]$. Then, we have*

$$\frac{\tau^2+1}{\tau+1} \leq \Gamma(\tau+1) \leq \frac{\tau^2+2}{\tau+2}. \tag{6}$$

Remark 2. *For $\tau \in [0,1]$, according to Proposition 4, we get*

$$\frac{1}{\Gamma(\tau+1)} \leq \frac{1}{2(\sqrt{2}-1)}. \tag{7}$$

Definition 1 ([11,44]). *Let $I \subset \mathbb{R}$.*

(a) The interval I is called a generalized interval if it is either an interval or $\{\rho_1\}$ or $\emptyset$.

(b) A partition of I is a finite set $\mathcal{P}$ such that each x in I lies in exactly one of the generalized intervals E in $\mathcal{P}$.

(c) A function $g : I \to \mathbb{R}$ is called piecewise constant with respect to the partition $\mathcal{P}$ of I if for any $E \in \mathcal{P}$, g is constant on E.

Theorem 1 ([45]). *(Arzela–Ascoli theorem) Let Λ be a subset of $C(\varpi, \mathbb{R})$. Λ is relatively compact if:*

1. *Λ is uniformly bounded.*
2. *Λ is équicontinuous.*

The following fixed point theorem will be used in the proof of our main results.

Theorem 2 ([6]). *(Schauder fixed point theorem) Let Λ be a convex subset of a Banach space E and $\mathcal{F} : \Lambda \longrightarrow \Lambda$ be a continuous and compact map. Then, $\mathcal{F}$ possesses a fixed point in Λ.*

Finally, we will extend the definition in [46] as follows:

Definition 2. *The BVP (1)–(3) is Ulam–Hyers (**UH**) stable if there exists $c_\Psi > 0$ such that for any $\epsilon > 0$ and for every solution $z \in PC^1(\varpi, \mathbb{R})$ satisfying*

$$\begin{cases} |{}^c D_{M_\vartheta^+}^{\tau(t)} z(t) - \Psi(t, z(t), I_{M_\vartheta^+}^{\tau(t)} z(t))| \leq \epsilon,\ t \in \varpi, \\ |\Delta x|_{t=M_\vartheta} - \Phi_\vartheta(x(M_\vartheta^-))| \leq \epsilon,\ \vartheta = 1, \ldots, n, \end{cases} \tag{8}$$

there exists a solution $x \in PC^1(\varpi, \mathbb{R})$ of the BVP (1)–(3), such that

$$|z(t) - x(t)| \leq c_\Psi \epsilon,\ t \in \varpi.$$

3. Main Existence and Uniqueness Results

Let us introduce the following assumption:

(A1) Let $\mathcal{P} = \{\varpi_0 := [M_0, M_1], \varpi_1 := (M_1, M_2], \varpi_2 := (M_2, M_3], \ldots, \varpi_n := (M_n, M_{n+1}]\}$ be a partition of the interval ϖ (with $M_0 = 1, M_{n+1} = M$) and let $\tau(t) : \varpi \to (0, 1]$ be a piecewise constant function with respect to $\mathcal{P}$ and $\tau^* = \sup_{t\in\varpi} \tau(t)$, i.e.,

$$\tau(t) = \sum_{\vartheta=0}^{n} \tau_\vartheta I_\vartheta(t) = \begin{cases} \tau_0, & if\ t \in \varpi_0, \\ \tau_1, & if\ t \in \varpi_1, \\ . \\ . \\ . \\ \tau_n, & if\ t \in \varpi_n, \end{cases}$$

where $0 < \tau_\vartheta \le \tau^* \le 1$ are constants, and

$$I_\vartheta(t) = \begin{cases} 1, & for\ t \in \varpi_\vartheta, \\ 0, & elsewhere. \end{cases} \quad \vartheta = 0, 1, \ldots, n.$$

In addition, we will give the definition of the solution to the BVP (1)–(3).

Definition 3. *The function $x \in PC(\varpi, \mathbb{R})$ is a solution of the BVP (1)–(3) if x fulfills the equation ${}^cD^{\tau(t)}_{M_\vartheta^+} x(t) = \Psi(t, x(t), I^{\tau(t)}_{M_\vartheta^+} x(t))$ on ϖ_ϑ and the conditions*

$$\Delta x|_{t=M_\vartheta} = \Phi_\vartheta(x(M_\vartheta^-)),\ \vartheta = 1, \ldots, n,$$

and

$$ax(1) + bx(M) = c.$$

First, we will analyze the Equation (1) of the BVP (1)–(3). For any $t \in (M_\vartheta, M_{\vartheta+1}]$, $\vartheta = 0, 1, \ldots, n$, it becomes a Caputo–Hadamard fractional differential equation of a variable order $\tau(t)$ for $x(t) \in C(\varpi, \mathbb{R})$, with CHFD given by (5). Then, for the sum, we have

$${}^cD^{\tau(t)}_{0^+} x(t) = \int_0^{M_1} \frac{(log\frac{t}{\varrho})^{-\tau_0}}{\Gamma(1-\tau_0)} x'(\varrho) d\varrho + \ldots + \int_{M_\vartheta}^{t} \frac{(log\frac{t}{\varrho})^{-\tau_\vartheta}}{\Gamma(1-\tau_\vartheta)} x'(\varrho) d\varrho. \tag{9}$$

Thus, according to (9), the Equation (1) can be written for any $t \in (M_\vartheta, M_{\vartheta+1}]$, $\vartheta = 0, 1, \ldots, n$ in the form

$$\int_0^{M_1} \frac{(log\frac{t}{\varrho})^{-\tau_0}}{\Gamma(1-\tau_0)} x'(\varrho) d\varrho + \ldots + \int_{M_\vartheta}^{t} \frac{(log\frac{t}{\varrho})^{-\tau_\vartheta}}{\Gamma(1-\tau_\vartheta)} x'(\varrho) d\varrho = \Psi(t, x(t), I^{\tau_\vartheta}_{0^+} x(t)). \tag{10}$$

In the case, when $x(t) \equiv 0$ on $t \in [0, M_\vartheta]/\{M_1, \ldots, M_{\vartheta-1}\}$, the Equation (1) is reduced to

$${}^cD^{\tau_\vartheta}_{M_\vartheta^+} x(t) = \Psi(t, x(t), I^{\tau_\vartheta}_{M_\vartheta^+} x(t)),\ t \in \varpi_\vartheta.$$

We need the following auxiliary proposition.

Proposition 5. *Let $\eta : \varpi \to \mathbb{R}$ be continuous. The solution of the following impulsive BVP*

$${}^cD^{\tau_\vartheta}_{M_\vartheta^+} x(t) = \eta(t),\ t \in \varpi_\vartheta, \tag{11}$$

$$\Delta x|_{t=M_\vartheta} = \Phi_\vartheta(x(M_\vartheta^-)), \quad \vartheta = 1, \ldots, n, \tag{12}$$

$$ax(1) + bx(M) = c, \tag{13}$$

is given by

$$x(t)=\begin{cases} \frac{-1}{a+b}\Big[b\sum_{s=1}^{n}\frac{1}{\Gamma(\tau_{s-1})}\int_{M_{s-1}}^{M_s}(\log\frac{M_s}{\varrho})^{\tau_{s-1}-1}\eta(\varrho)\frac{d\varrho}{\varrho} \\ +\frac{b}{\Gamma(\tau_n)}\int_{M_n}^{M}(\log\frac{M}{\varrho})^{\tau_n-1}\eta(\varrho)\frac{d\varrho}{\varrho} \\ +b\sum_{s=1}^{n}\Phi_s(x(M_s^-))-c\Big]+\frac{1}{\Gamma(\tau_0)}\int_1^t(\log\frac{t}{\varrho})^{\tau_0-1}\eta(\varrho)\frac{d\varrho}{\varrho}, \qquad t\in[M_0,M_1], \\ \frac{-1}{a+b}\Big[b\sum_{s=1}^{n}\frac{1}{\Gamma(\tau_{s-1})}\int_{M_{s-1}}^{M_s}(\log\frac{M_s}{\varrho})^{\tau_{s-1}-1}\eta(\varrho)\frac{d\varrho}{\varrho} \\ +\frac{b}{\Gamma(\tau_n)}\int_{M_n}^{M}(\log\frac{M}{\varrho})^{\tau_n-1}\eta(\varrho)\frac{d\varrho}{\varrho} \\ +b\sum_{s=1}^{n}\Phi_s(x(M_s^-))-c\Big]+\sum_{s=1}^{\vartheta}\frac{1}{\Gamma(\tau_{s-1})}\int_{M_{s-1}}^{M_s}(\log\frac{M_s}{\varrho})^{\tau_{s-1}-1}\eta(\varrho)\frac{d\varrho}{\varrho} \\ +\frac{1}{\Gamma(\tau_\vartheta)}\int_{M_\vartheta}^{t}(\log\frac{t}{\varrho})^{\tau_\vartheta-1}\eta(\varrho)\frac{d\varrho}{\varrho}+\sum_{s=1}^{\vartheta}\Phi_s(x(M_s^-)), \quad t\in(M_\vartheta,M_{\vartheta+1}],\ \vartheta=1,\dots,n. \end{cases} \tag{14}$$

Proof. Let x be a solution of the BVP (11)–(13). If $t\in[M_0,M_1]$, by Proposition 1, we get

$$x(t)=\omega_0+\frac{1}{\Gamma(\tau_0)}\int_1^t(\log\frac{t}{\varrho})^{\tau_0-1}\eta(\varrho)\frac{d\varrho}{\varrho}, \quad \omega_0\in\mathbb{R}.$$

If $t\in(M_1,M_2]$, then Proposition 1 implies

$$\begin{aligned} x(t) &= x(M_1^+)+\frac{1}{\Gamma(\tau_1)}\int_{M_1}^{t}(\log\frac{t}{\varrho})^{\tau_1-1}\eta(\varrho)\frac{d\varrho}{\varrho} \\ &= \Delta x|_{t=M_1}+x(M_1^-)+\frac{1}{\Gamma(\tau_1)}\int_{M_1}^{t}(\log\frac{t}{\varrho})^{\tau_1-1}\eta(\varrho)\frac{d\varrho}{\varrho} \\ &= \omega_0+\Phi_1(x(M_1^-))+\frac{1}{\Gamma(\tau_0)}\int_1^{M_1}(\log\frac{M_1}{\varrho})^{\tau_0-1}\eta(\varrho)\frac{d\varrho}{\varrho}+\frac{1}{\Gamma(\tau_1)}\int_{M_1}^{t}(\log\frac{t}{\varrho})^{\tau_1-1}\eta(\varrho)\frac{d\varrho}{\varrho}. \end{aligned}$$

If $t\in(M_2,M_3]$, by Proposition 1, we get

$$\begin{aligned} x(t) &= x(M_2^+)+\frac{1}{\Gamma(\tau_2)}\int_{M_2}^{t}(\log\frac{t}{\varrho})^{\tau_2-1}\eta(\varrho)\frac{d\varrho}{\varrho} \\ &= \Delta x|_{t=M_2}+x(M_2^-)+\frac{1}{\Gamma(\tau_2)}\int_{M_2}^{t}(\log\frac{t}{\varrho})^{\tau_2-1}\eta(\varrho)\frac{d\varrho}{\varrho} \\ &= \omega_0+\Phi_2(x(M_2^-))+\Phi_1(x(M_1^-))+\frac{1}{\Gamma(\tau_0)}\int_1^{M_1}(\log\frac{M_1}{\varrho})^{\tau_0-1}\eta(\varrho)\frac{d\varrho}{\varrho} \\ &+ \frac{1}{\Gamma(\tau_1)}\int_{M_1}^{M_2}(\log\frac{M_2}{\varrho})^{\tau_1-1}\eta(\varrho)\frac{d\varrho}{\varrho}+\frac{1}{\Gamma(\tau_2)}\int_{M_2}^{t}(\log\frac{t}{\varrho})^{\tau_2-1}\eta(\varrho)\frac{d\varrho}{\varrho}. \end{aligned}$$

Then, the solution $x(t)$ for $t\in(M_\vartheta,M_{\vartheta+1}]$, $\vartheta=1,\dots,n$, can be written as

$$\begin{aligned} x(t) &= \omega_0+\sum_{s=1}^{\vartheta}\frac{1}{\Gamma(\tau_{s-1})}\int_{M_{s-1}}^{M_s}(\log\frac{M_s}{\varrho})^{\tau_{s-1}-1}\eta(\varrho)\frac{d\varrho}{\varrho} \\ &\quad+\frac{1}{\Gamma(\tau_\vartheta)}\int_{M_\vartheta}^{t}(\log\frac{t}{\varrho})^{\tau_\vartheta-1}\eta(\varrho)\frac{d\varrho}{\varrho}+\sum_{s=1}^{\vartheta}\Phi_s(x(M_s^-)). \end{aligned}$$

Applying the boundary conditions $ax(1)+bx(M)=c$, we have

$$\begin{aligned} c &= \omega_0(a+b)+b\sum_{s=1}^{n}\frac{1}{\Gamma(\tau_{s-1})}\int_{M_{s-1}}^{M_s}(\log\frac{M_s}{\varrho})^{\tau_{s-1}-1}\eta(\varrho)\frac{d\varrho}{\varrho} \\ &\quad+\frac{b}{\Gamma(\tau_n)}\int_{M_n}^{M}(\log\frac{M}{\varrho})^{\tau_n-1}\eta(\varrho)\frac{d\varrho}{\varrho}+b\sum_{s=1}^{n}\Phi_s(x(M_s^-)). \end{aligned}$$

Then,

$$\begin{aligned}\omega_0 &= \frac{-1}{a+b}\Big[b\sum_{s=1}^{n}\frac{1}{\Gamma(\tau_{s-1})}\int_{M_{s-1}}^{M_s}(\log\frac{M_s}{\varrho})^{\tau_{s-1}-1}\eta(\varrho)\frac{d\varrho}{\varrho}\\ &+\frac{b}{\Gamma(\tau_n)}\int_{M_n}^{M}(\log\frac{M}{\varrho})^{\tau_n-1}\eta(\varrho)\frac{d\varrho}{\varrho}\\ &+b\sum_{s=1}^{n}\Phi_s(x(M_s^-))-c\Big].\end{aligned}$$

Thus,

$$\begin{aligned}x(t) &= \frac{-1}{a+b}\Big[b\sum_{s=1}^{n}\frac{1}{\Gamma(\tau_{s-1})}\int_{M_{s-1}}^{M_s}(\log\frac{M_s}{\varrho})^{\tau_{s-1}-1}\eta(\varrho)\frac{d\varrho}{\varrho}+\frac{b}{\Gamma(\tau_n)}\int_{M_n}^{M}(\log\frac{M}{\varrho})^{\tau_n-1}\eta(\varrho)\frac{d\varrho}{\varrho}\\ &+b\sum_{s=1}^{n}\Phi_s(x(M_s^-))-c\Big]+\sum_{s=1}^{\vartheta}\frac{1}{\Gamma(\tau_{s-1})}\int_{M_{s-1}}^{M_s}(\log\frac{M_s}{\varrho})^{\tau_{s-1}-1}\eta(\varrho)\frac{d\varrho}{\varrho}\\ &+\frac{1}{\Gamma(\tau_\vartheta)}\int_{M_\vartheta}^{t}(\log\frac{t}{\varrho})^{\tau_\vartheta-1}\eta(\varrho)\frac{d\varrho}{\varrho}+\sum_{s=1}^{\vartheta}\Phi_s(x(M_s^-)).\end{aligned}$$

Conversely, we can easily show that x solves the BVP (11)–(12). □

Now, we present our first result, assuming that the following assumptions are satisfied:

(A2) For $0\le\gamma\le\min_{t\in\varpi}|\tau(t)|$, the function $t^\gamma\Psi:\varpi\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is continuous and there exist constants $D_1,D_2>0$, such that, $t^\gamma|\Psi(t,x_1,y_1)-\Psi(t,x_2,y_2)|\le D_1|x_1-x_2|+D_2|y_1-y_2|$, for any $x_1,\ x_2,\ y_1,\ y_2\in\mathbb{R}$ and $t\in\varpi$.

(A3) For any $\vartheta=1,\dots,n$, $x\in\mathbb{R}$ and $t\in\varpi$, there exists $D_3>0$ such that,

$$|\Phi_\vartheta(x(t))|\le D_3|x(t)|.$$

Theorem 3. *Let conditions (A1)–(A3) hold, and*

$$\Big(\frac{|b|}{|a+b|}+1\Big)\Big[\frac{(n+1)(\log M)^{\tau^*}}{2(\sqrt{2}-1)}\Big(D_1+D_2\frac{(\log M)^{\tau^*}}{2(\sqrt{2}-1)}\Big)+nD_3\Big]<1. \quad (15)$$

Then, the BVP (1)–(3) possesses a solution on $PC(\varpi,\mathbb{R})$.

Proof. We construct the operator

$$S:PC(\varpi,\mathbb{R})\to PC(\varpi,\mathbb{R}),$$

as follow

$$\begin{aligned}Sx(t) &= \frac{-1}{a+b}\Big[b\sum_{s=1}^{n}\frac{1}{\Gamma(\tau_{s-1})}\int_{M_{s-1}}^{M_s}(\log\frac{M_s}{\varrho})^{\tau_{s-1}-1}\Psi(\varrho,x(\varrho),I_{M_{s-1}^+}^{\tau_{s-1}}x(\varrho))\frac{d\varrho}{\varrho}\\ &+\frac{b}{\Gamma(\tau_n)}\int_{M_n}^{M}(\log\frac{M}{\varrho})^{\tau_n-1}\Psi(\varrho,x(\varrho),I_{M_n^+}^{\tau_n}x(\varrho))\frac{d\varrho}{\varrho}+b\sum_{s=1}^{n}\Phi_s(x(M_s^-))-c\Big]\\ &+\sum_{0<M_s<t}\frac{1}{\Gamma(\tau_{s-1})}\int_{M_{s-1}}^{M_s}(\log\frac{M_s}{\varrho})^{\tau_{s-1}-1}\Psi(\varrho,x(\varrho),I_{M_{s-1}^+}^{\tau_{s-1}}x(\varrho))\frac{d\varrho}{\varrho}\\ &+\frac{1}{\Gamma(\tau_\vartheta)}\int_{M_\vartheta}^{t}(\log\frac{t}{\varrho})^{\tau_\vartheta-1}\Psi(\varrho,x(\varrho),I_{M_\vartheta^+}^{\tau_\vartheta}x(\varrho))\frac{d\varrho}{\varrho}\\ &+\sum_{0<M_s<t}\Phi_s(x(M_s^-)).\end{aligned} \quad (16)$$

The operator S defined in (16) is well defined from the continuity of function $t^{\gamma}\Psi$ and from the properties of fractional integrals.

Let the set

$$B_R = \{x \in PC(\varpi, \mathbb{R}),\ \|x\| \le R\},$$

where

$$R \ge \frac{(\frac{|b|}{|a+b|}+1)\frac{(n+1)(\log M)^{\tau^*}}{2(\sqrt{2}-1)}\Psi^\star + \frac{|c|}{|a+b|}}{1-\left(\frac{|b|}{|a+b|}+1\right)\left[\frac{(n+1)(\log M)^{\tau^*}}{2(\sqrt{2}-1)}\left(D_1+D_2\frac{(\log M)^{\tau^*}}{2(\sqrt{2}-1)}\right)+nD_3\right]},$$

and

$$\Psi^\star = \sup_{t\in\varpi} |\Psi(t,0,0)|.$$

Clearly B_R is non-empty, closed, convex, and bounded.

Step 1: Claim: $S(B_R) \subseteq (B_R)$.

For $x \in B_R$, we get

$$|Sx(t)|$$

$$\begin{aligned}
\le\ & \frac{1}{|a+b|}\Big[|b|\sum_{s=1}^{n}\frac{1}{\Gamma(\tau_{s-1})}\int_{M_{s-1}}^{M_s}(\log\frac{M_s}{\varrho})^{\tau_{s-1}-1}\Big|\Psi(\varrho,x(\varrho),I^{\tau_{s-1}}_{M^+_{s-1}}x(\varrho))-\Psi(\varrho,0,0)\Big|\frac{d\varrho}{\varrho}\\
&+|b|\sum_{s=1}^{n}\frac{1}{\Gamma(\tau_{s-1})}\int_{M_{s-1}}^{M_s}(\log\frac{M_s}{\varrho})^{\tau_{s-1}-1}\Big|\Psi(\varrho,0,0)\Big|\frac{d\varrho}{\varrho}\\
&+\frac{|b|}{\Gamma(\tau_n)}\int_{M_n}^{M}(\log\frac{M}{\varrho})^{\tau_n-1}\Big|\Psi(\varrho,x(\varrho),I^{\tau_n}_{M^+_n}x(\varrho))-\Psi(\varrho,0,0)\Big|\frac{d\varrho}{\varrho}+\frac{|b|}{\Gamma(\tau_n)}\int_{M_n}^{M}(\log\frac{M}{\varrho})^{\tau_n-1}\Big|\Psi(\varrho,0,0)\Big|\frac{d\varrho}{\varrho}\\
&+|b|\sum_{s=1}^{n}\Big|\Phi_s(x(M_s^-))\Big|+|c|\Big]\\
&+\sum_{0<M_s<t}\frac{1}{\Gamma(\tau_{s-1})}\int_{M_{s-1}}^{M_s}(\log\frac{M_s}{\varrho})^{\tau_{s-1}-1}\Big|\Psi(\varrho,x(\varrho),I^{\tau_{s-1}}_{M^+_{s-1}}x(\varrho))-\Psi(\varrho,0,0)\Big|\frac{d\varrho}{\varrho}\\
&+\sum_{0<M_s<t}\frac{1}{\Gamma(\tau_{s-1})}\int_{M_{s-1}}^{M_s}(\log\frac{M_s}{\varrho})^{\tau_{s-1}-1}\Big|\Psi(\varrho,0,0)\Big|\frac{d\varrho}{\varrho}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\tau_\vartheta)}\int_{M_\vartheta}^{t}(\log\frac{t}{\varrho})^{\tau_\vartheta-1}\Big|\Psi(\varrho,x(\varrho),I_{M_\vartheta^+}^{\tau_\vartheta}x(\varrho))-\Psi(\varrho,0,0)\Big|\frac{d\varrho}{\varrho}+\frac{1}{\Gamma(\tau_\vartheta)}\int_{M_\vartheta}^{t}(\log\frac{t}{\varrho})^{\tau_\vartheta-1}\Big|\Psi(\varrho,0,0)\Big|\frac{d\varrho}{\varrho}\\
&+\sum_{0<M_s<t}\Big|\Phi_s(x(M_s^-))\Big|\\
\leq\;&\frac{1}{|a+b|}\Big[|b|\sum_{s=1}^{n}\frac{1}{\Gamma(\tau_{s-1})}\int_{M_{s-1}}^{M_s}(\log\frac{M_s}{\varrho})^{\tau_{s-1}-1}\Big(D_1|x(\varrho)|+D_2|I_{M_{s-1}^+}^{\tau_{s-1}}x(\varrho)|\Big)\frac{d\varrho}{\varrho}\\
&+|b|\Psi^\star\sum_{s=1}^{n}\frac{1}{\Gamma(\tau_{s-1})}\int_{M_{s-1}}^{M_s}(\log\frac{M_s}{\varrho})^{\tau_{s-1}-1}\frac{d\varrho}{\varrho}+\frac{|b|}{\Gamma(\tau_n)}\int_{M_n}^{M}(\log\frac{M}{\varrho})^{\tau_n-1}\Big(D_1|x(\varrho)|+D_2|I_{M_n^+}^{\tau_n}x(\varrho)|\Big)\frac{d\varrho}{\varrho}\\
&+\frac{|b|\Psi^\star}{\Gamma(\tau_n)}\int_{M_n}^{M}(\log\frac{M}{\varrho})^{\tau_n-1}\frac{d\varrho}{\varrho}+|b|\sum_{s=1}^{n}D_3\Big|x(M_s^-)\Big|+|c|\Big]\\
&+\sum_{0<M_s<t}\frac{1}{\Gamma(\tau_{s-1})}\int_{M_{s-1}}^{M_s}(\log\frac{M_s}{\varrho})^{\tau_{s-1}-1}\Big(D_1|x(\varrho)|+D_2|I_{M_{s-1}^+}^{\tau_{s-1}}x(\varrho)|\Big)\frac{d\varrho}{\varrho}\\
&+\sum_{0<M_s<t}\frac{\Psi^\star}{\Gamma(\tau_{s-1})}\int_{M_{s-1}}^{M_s}(\log\frac{M_s}{\varrho})^{\tau_{s-1}-1}\frac{d\varrho}{\varrho}+\frac{1}{\Gamma(\tau_\vartheta)}\int_{M_\vartheta}^{t}(\log\frac{t}{\varrho})^{\tau_\vartheta-1}\Big(D_1|x(\varrho)|+D_2|I_{M_\vartheta^+}^{\tau_\vartheta}x(\varrho)|\Big)\frac{d\varrho}{\varrho}\\
&+\frac{\Psi^\star}{\Gamma(\tau_\vartheta)}\int_{M_\vartheta}^{t}(\log\frac{t}{\varrho})^{\tau_\vartheta-1}\frac{d\varrho}{\varrho}+\sum_{0<M_s<t}D_3\Big|x(M_s^-)\Big|\\
\leq\;&\frac{1}{|a+b|}\Big[|b|\sum_{s=1}^{n}\frac{1}{\Gamma(\tau_{s-1})}\int_{M_{s-1}}^{M_s}(\log\frac{M_s}{\varrho})^{\tau_{s-1}-1}\Big(D_1\|x\|+D_2\|\frac{(\log M)^{\tau_{s-1}}}{\Gamma(\tau_{s-1}+1)}x\|\Big)\frac{d\varrho}{\varrho}\\
&+|b|\Psi^\star\sum_{s=1}^{n}\frac{1}{\Gamma(\tau_{s-1})}\int_{M_{s-1}}^{M_s}(\log\frac{M_s}{\varrho})^{\tau_{s-1}-1}\frac{d\varrho}{\varrho}+\frac{|b|}{\Gamma(\tau_n)}\int_{M_n}^{M}(\log\frac{M}{\varrho})^{\tau_n-1}\Big(D_1\|x\|+D_2\|\frac{(\log M)^{\tau_n}}{\Gamma(\tau_n+1)}x\|\Big)\frac{d\varrho}{\varrho}\\
&+\frac{|b|\Psi^\star}{\Gamma(\tau_n)}\int_{M_n}^{M}(\log\frac{M}{\varrho})^{\tau_n-1}\frac{d\varrho}{\varrho}+|b|\sum_{s=1}^{n}D_3\|x\|+|c|\Big]\\
&+\sum_{s=1}^{n}\frac{1}{\Gamma(\tau_{s-1})}\int_{M_{s-1}}^{M_s}(\log\frac{M_s}{\varrho})^{\tau_{s-1}-1}\Big(D_1\|x\|+D_2\|\frac{(\log M)^{\tau_{s-1}}}{\Gamma(\tau_{s-1}+1)}x\|\Big)\frac{d\varrho}{\varrho}\\
&+\sum_{s=1}^{n}\frac{\Psi^\star}{\Gamma(\tau_{s-1})}\int_{M_{s-1}}^{M_s}(\log\frac{M_s}{\varrho})^{\tau_{s-1}-1}\frac{d\varrho}{\varrho}+\frac{1}{\Gamma(\tau_\vartheta)}\int_{M_\vartheta}^{t}(\log\frac{t}{\varrho})^{\tau_\vartheta-1}\Big(D_1\|x\|+D_2\|\frac{(\log M)^{\tau_\vartheta}}{\Gamma(\tau_\vartheta+1)}x\|\Big)\frac{d\varrho}{\varrho}\\
&+\frac{\Psi^\star}{\Gamma(\tau_\vartheta)}\int_{M_\vartheta}^{t}(\log\frac{t}{\varrho})^{\tau_\vartheta-1}\frac{d\varrho}{\varrho}+\sum_{s=1}^{n}D_3\|x\|\\
\leq\;&\frac{1}{|a+b|}\Big[|b|\frac{n(\log M)^{\tau_{s-1}}}{\Gamma(\tau_{s-1}+1)}\Big(D_1+D_2\frac{(\log M)^{\tau_{s-1}}}{\Gamma(\tau_{s-1}+1)}\Big)\|x\|+|b|\Psi^\star\frac{n(logM)^{\tau_{s-1}}}{\Gamma(\tau_{s-1}+1)}\\
&+\frac{|b|(\log M)^{\tau_n}}{\Gamma(\tau_n+1)}\Big(D_1+D_2\frac{(\log M)^{\tau_n}}{\Gamma(\tau_n+1)}\Big)\|x\|+\frac{|b|\Psi^\star(\log M)^{\tau_n}}{\Gamma(\tau_n+1)}+|b|nD_3\|x\|+|c|\Big]\\
&+\frac{n(\log M)^{\tau_{s-1}}}{\Gamma(\tau_{s-1}+1)}\Big(D_1+D_2\frac{(\log M)^{\tau_{s-1}}}{\Gamma(\tau_{s-1}+1)}\Big)\|x\|+\frac{n\Psi^\star(\log M)^{\tau_{s-1}}}{\Gamma(\tau_{s-1}+1)}\\
&+\frac{(\log M)^{\tau_\vartheta}}{\Gamma(\tau_\vartheta+1)}\Big(D_1+D_2\frac{(\log M)^{\tau_\vartheta}}{\Gamma(\tau_\vartheta+1)}\Big)\|x\|+\frac{\Psi^\star(\log M)^{\tau_\vartheta}}{\Gamma(\tau_\vartheta+1)}+nD_3\|x\|\\
\leq\;&\Big(\frac{|b|}{|a+b|}+1\Big)\Big[\frac{(n+1)(\log M)^{\tau^*}}{2(\sqrt{2}-1)}\Big(D_1+D_2\frac{(\log M)^{\tau^*}}{2(\sqrt{2}-1)}\Big)+nD_3\Big]\|x\|\\
&+(\frac{|b|}{|a+b|}+1)\frac{(n+1)(\log M)^{\tau^*}}{2(\sqrt{2}-1)}\Psi^\star+\frac{|c|}{|a+b|}\\
\leq\;&R.
\end{aligned}
$$

Step 2: Claim: S is continuous.

Let the sequence (x_n) converges to x in $PC(\varpi,\mathbb{R})$. We will prove that

$$\|Sx_n - Sx\| \to 0.$$

For $t \in \varpi$, we have

$$|Sx_n(t) - Sx(t)|$$

$$\begin{aligned}
\leq\ & \frac{1}{|a+b|}\Big[|b|\sum_{s=1}^{n}\frac{1}{\Gamma(\tau_{s-1})}\int_{M_{s-1}}^{M_s}(\log\frac{M_s}{\varrho})^{\tau_{s-1}-1}\Big|\Psi(\varrho,x_n(\varrho),I^{\tau_{s-1}}_{M^+_{s-1}}x_n(\varrho))-\Psi(\varrho,x(\varrho),I^{\tau_{s-1}}_{M^+_{s-1}}x(\varrho))\Big|\frac{d\varrho}{\varrho}\\
&+\frac{|b|}{\Gamma(\tau_n)}\int_{M_n}^{M}(\log\frac{M}{\varrho})^{\tau_n-1}\Big|\Psi(\varrho,x_n(\varrho),I^{\tau_n}_{M^+_n}x_n(\varrho))-\Psi(\varrho,x(\varrho),I^{\tau_n}_{M^+_n}x(\varrho))\Big|\frac{d\varrho}{\varrho}\\
&+|b|\sum_{s=1}^{n}\Big|\Phi_s(x_n(M_s^-))-\Phi_s(x(M_s^-))\Big|\Big]\\
&+\sum_{0<M_s<t}\frac{1}{\Gamma(\tau_{s-1})}\int_{M_{s-1}}^{M_s}(\log\frac{M_s}{\varrho})^{\tau_{s-1}-1}\Big|\Psi(\varrho,x_n(\varrho),I^{\tau_{s-1}}_{M^+_{s-1}}x_n(\varrho))-\Psi(\varrho,x(\varrho),I^{\tau_{s-1}}_{M^+_{s-1}}x(\varrho))\Big|\frac{d\varrho}{\varrho}\\
&+\frac{1}{\Gamma(\tau_\vartheta)}\int_{M_\vartheta}^{t}(\log\frac{t}{\varrho})^{\tau_\vartheta-1}\Big|\Psi(\varrho,x_n(\varrho),I^{\tau_\vartheta}_{M^+_\vartheta}x_n(\varrho))-\Psi(\varrho,x(\varrho),I^{\tau_\vartheta}_{M^+_\vartheta}x(\varrho))\Big|\frac{d\varrho}{\varrho}\\
&+\sum_{0<M_s<t}\Big|\Phi_s(x_n(M_s^-))-\Phi_s(x(M_s^-))\Big|\\
\leq\ & \frac{1}{|a+b|}\Big[|b|\sum_{s=1}^{n}\frac{1}{\Gamma(\tau_{s-1})}\int_{M_{s-1}}^{M_s}(\log\frac{M_s}{\varrho})^{\tau_{s-1}-1}\Big(D_1|x_n(\varrho)-x(\varrho)|+D_2|I^{\tau_{s-1}}_{M^+_{s-1}}(x_n(\varrho)-x(\varrho))|\Big)\frac{d\varrho}{\varrho}\\
&+\frac{|b|}{\Gamma(\tau_n)}\int_{M_n}^{M}(\log\frac{M}{\varrho})^{\tau_n-1}\Big(D_1|x_n(\varrho)-x(\varrho)|+D_2|I^{\tau_n}_{M^+_n}(x_n(\varrho)-x(\varrho))|\Big)\frac{d\varrho}{\varrho}\Big]\\
&+\sum_{s=1}^{n}\frac{1}{\Gamma(\tau_{s-1})}\int_{M_{s-1}}^{M_s}(\log\frac{M_s}{\varrho})^{\tau_{s-1}-1}\Big(D_1|x_n(\varrho)-x(\varrho)|+D_2|I^{\tau_{s-1}}_{M^+_{s-1}}(x_n(\varrho)-x(\varrho))|\Big)\frac{d\varrho}{\varrho}\\
&+\frac{1}{\Gamma(\tau_\vartheta)}\int_{M_\vartheta}^{t}(\log\frac{t}{\varrho})^{\tau_\vartheta-1}\Big(D_1|x_n(\varrho)-x(\varrho)|+D_2|I^{\tau_\vartheta}_{M^+_\vartheta}(x_n(\varrho)-x(\varrho))|\Big)\frac{d\varrho}{\varrho}\\
&+\Big(\frac{|b|}{|a+b|}+1\Big)\sum_{s=1}^{n}\Big|\Phi_s(x_n(M_s^-))-\Phi_s(x(M_s^-))\Big|\\
\leq\ & \frac{1}{|a+b|}\Big[|b|\frac{n(\log M)^{\tau_{s-1}}}{\Gamma(\tau_{s-1}+1)}\Big(D_1+D_2\frac{(\log M)^{\tau_{s-1}}}{\Gamma(\tau_{s-1}+1)}\Big)\|x_n-x\|\\
&+\frac{|b|(\log M)^{\tau_n}}{\Gamma(\tau_n+1)}\Big(D_1+D_2\frac{(\log M)^{\tau_n}}{\Gamma(\tau_n+1)}\Big)\|x_n-x\|\Big]+\frac{n(\log M)^{\tau_{s-1}}}{\Gamma(\tau_{s-1}+1)}\Big(D_1+D_2\frac{(\log M)^{\tau_{s-1}}}{\Gamma(\tau_{s-1}+1)}\Big)\|x_n-x\|\\
&+\frac{(\log M)^{\tau_\vartheta}}{\Gamma(\tau_\vartheta+1)}\Big(D_1+D_2\frac{(\log M)^{\tau_\vartheta}}{\Gamma(\tau_\vartheta+1)}\Big)\|x_n-x\|\\
\leq\ & \Big(\frac{|b|}{|a+b|}+1\Big)\Big(\frac{(n+1)(\log M)^{\tau^*}}{2(\sqrt{2}-1)}\Big)\Big(D_1+D_2\frac{(\log M)^{\tau^*}}{2(\sqrt{2}-1)}\Big)\|x_n-x\|\\
&+\Big(\frac{|b|}{|a+b|}+1\Big)\sum_{s=1}^{n}\Big|\Phi_s(x_n(M_s^-))-\Phi_s(x(M_s^-))\Big|.
\end{aligned}$$

Since Φ_s is continuous, then

$$\|Sx_n - Sx\| \to 0 \;\; as \;\; n \to \infty.$$

Then, S is continuous.

Step 3: Claim: S is compact.

By Step 1, we have $\|S(x)\| \leq R$ for each $x \in B_R$, which means that $S(B_R)$ is bounded. Now we will show that $S(B_R)$ is equicontinuous.

For $t_1, t_2 \in \varpi,\ t_1 < t_2$ and $x \in B_R$, we have

$$
\begin{aligned}
&|Sx(t_2)-Sx(t_1)| \\
\leq\ & \frac{1}{\Gamma(\tau_\vartheta)}\int_1^{t_1}\left|(\log\frac{t_2}{\varrho})^{\tau_\vartheta-1}-(\log\frac{t_1}{\varrho})^{\tau_\vartheta-1}\right||\Psi(\varrho,x(\varrho),I^{\tau_\vartheta}_{M^+_\vartheta}x(\varrho))|\frac{d\varrho}{\varrho} \\
&+\frac{1}{\Gamma(\tau_\vartheta)}\int_{t_1}^{t_2}(\log\frac{t_2}{\varrho})^{\tau_\vartheta-1}|\Psi(\varrho,x(\varrho),I^{\tau_\vartheta}_{M^+_\vartheta}x(\varrho))|\frac{d\varrho}{\varrho}+\sum_{0<M_s<t_2-t_1}|\Phi_s(x(M_s^-))| \\
\leq\ & \frac{1}{\Gamma(\tau_\vartheta)}\int_1^{t_1}\left|(\log\frac{t_2}{\varrho})^{\tau_\vartheta-1}-(\log\frac{t_1}{\varrho})^{\tau_\vartheta-1}\right||\Psi(\varrho,x(\varrho),I^{\tau_\vartheta}_{M^+_\vartheta}x(\varrho))-\Psi(\varrho,0,0)|\frac{d\varrho}{\varrho} \\
&+\frac{1}{\Gamma(\tau_\vartheta)}\int_1^{t_1}\left|(\log\frac{t_2}{\varrho})^{\tau_\vartheta-1}-(\log\frac{t_1}{\varrho})^{\tau_\vartheta-1}\right||\Psi(\varrho,0,0)|\frac{d\varrho}{\varrho} \\
&+\frac{1}{\Gamma(\tau_\vartheta)}\int_{t_1}^{t_2}(\log\frac{t_2}{\varrho})^{\tau_\vartheta-1}|\Psi(\varrho,x(\varrho),I^{\tau_\vartheta}_{M^+_\vartheta}x(\varrho))-\Psi(\varrho,0,0)|\frac{d\varrho}{\varrho}+\frac{1}{\Gamma(\tau_\vartheta)}\int_{t_1}^{t_2}(\log\frac{t_2}{\varrho})^{\tau_\vartheta-1}|\Psi(\varrho,0,0)|\frac{d\varrho}{\varrho} \\
&+\sum_{0<M_s<t_2-t_1}|\Phi_s(x(M_s^-))| \\
\leq\ & \frac{1}{\Gamma(\tau_\vartheta)}\int_1^{t_1}\left|(\log\frac{t_2}{\varrho})^{\tau_\vartheta-1}-(\log\frac{t_1}{\varrho})^{\tau_\vartheta-1}\right|(D_1|x(\varrho)|+D_2|I^{\tau_\vartheta}_{M^+_\vartheta}x(\varrho)|)\frac{d\varrho}{\varrho} \\
&+\frac{\Psi^\star}{\Gamma(\tau_\vartheta)}\int_1^{t_1}\left|(\log\frac{t_2}{\varrho})^{\tau_\vartheta-1}-(\log\frac{t_1}{\varrho})^{\tau_\vartheta-1}\right|\frac{d\varrho}{\varrho} \\
&+\frac{1}{\Gamma(\tau_\vartheta)}\int_{t_1}^{t_2}(\log\frac{t_2}{\varrho})^{\tau_\vartheta-1}(D_1|x(\varrho)|+D_2|I^{\tau_\vartheta}_{M^+_\vartheta}x(\varrho)|)\frac{d\varrho}{\varrho}+\frac{\Psi^\star}{\Gamma(\tau_\vartheta)}\int_{t_1}^{t_2}(\log\frac{t_2}{\varrho})^{\tau_\vartheta-1}\frac{d\varrho}{\varrho} \\
&+\sum_{0<M_s<t_2-t_1}|\Phi_s(x(M_s^-))| \\
\leq\ & \frac{1}{\Gamma(\tau_\vartheta)}\int_1^{t_1}\left|(\log\frac{t_2}{\varrho})^{\tau_\vartheta-1}-(\log\frac{t_1}{\varrho})^{\tau_\vartheta-1}\right|\Big(D_1\|x\|+D_2\|\frac{(\log M)^{\tau_\vartheta}}{\Gamma(\tau_\vartheta+1)}x\|\Big)\frac{d\varrho}{\varrho} \\
&+\frac{\Psi^\star}{\Gamma(\tau_\vartheta)}\int_1^{t_1}\left|(\log\frac{t_2}{\varrho})^{\tau_\vartheta-1}-(\log\frac{t_1}{\varrho})^{\tau_\vartheta-1}\right|\frac{d\varrho}{\varrho} \\
&+\frac{1}{\Gamma(\tau_\vartheta)}\int_{t_1}^{t_2}(\log\frac{t_2}{\varrho})^{\tau_\vartheta-1}\Big(D_1\|x\|+D_2\|\frac{(\log M)^{\tau_\vartheta}}{\Gamma(\tau_\vartheta+1)}x\|\Big)\frac{d\varrho}{\varrho}+\frac{\Psi^\star}{\Gamma(\tau_\vartheta)}\int_{t_1}^{t_2}(\log\frac{t_2}{\varrho})^{\tau_\vartheta-1}\frac{d\varrho}{\varrho} \\
&+\sum_{0<M_s<t_2-t_1}|\Phi_s(x(M_s^-))| \\
\leq\ & \frac{1}{\Gamma(\tau_\vartheta)}\Big[\Big(D_1+D_2\frac{(\log M)^{\tau_\vartheta}}{\Gamma(\tau_\vartheta+1)}\Big)\|x\|+\Psi^\star\Big]\int_1^{t_1}\left|(\log\frac{t_2}{\varrho})^{\tau_\vartheta-1}-(\log\frac{t_1}{\varrho})^{\tau_\vartheta-1}\right|\frac{d\varrho}{\varrho} \\
&+\frac{1}{\Gamma(\tau_\vartheta)}\Big[\Big(D_1+D_2\frac{(\log M)^{\tau_\vartheta}}{\Gamma(\tau_\vartheta+1)}\Big)\|x\|+\Psi^\star\Big]\int_{t_1}^{t_2}(\log\frac{t_2}{\varrho})^{\tau_\vartheta-1}\frac{d\varrho}{\varrho} \\
&+\sum_{0<M_s<t_2-t_1}|\Phi_s(x(M_s^-))|.
\end{aligned}
$$

As $t_1 \to t_2$, the right-hand side of the above inequality tends to zero. Hence $|Sx(t_2) - Sx(t_1)| \to 0$. It implies that $S(B_R)$ is equicontinuous.

Thus, by Theorem 2 the BVP (1)–(3) possesses a solution in B_R. Since $B_R \subset PC(\varpi, \mathbb{R})$ the claim of Theorem 3 is proved. □

Now, we will invoke the Banach contraction principle to verify the uniqueness of solutions for the BVP (1)–(3).

Theorem 4. *In addition to (A1) and (A2), assume that:*

(A4) *For $s = 1, \dots, n$, there exists $D_4 > 0$ such that, for any $x, y \in \mathbb{R}$ and $t \in \varpi$,* $|\Phi_s(x(t)) - \Phi_s(y(t))| \le D_4|x(t) - y(t)|$.

Then, if

$$\Big(\frac{|b|}{|a+b|}+1\Big)\Big[\frac{(n+1)(\log M)^{\tau^*}}{2(\sqrt{2}-1)}\Big(D_1+D_2\frac{(\log M)^{\tau^*}}{2(\sqrt{2}-1)}\Big)+nD_4\Big]<1, \tag{17}$$

the BVP (1)–(3) possesses a solution uniquely determined on $PC(\varpi,\mathbb{R})$.

Proof. For $t\in\varpi$ and $x,y\in PC(\varpi,\mathbb{R})$, we have

$$
\begin{aligned}
&|Sx(t)-Sy(t)|\\
\le\;& \frac{1}{|a+b|}\Big[|b|\sum_{s=1}^{n}\frac{1}{\Gamma(\tau_{s-1})}\int_{M_{s-1}}^{M_s}(\log\frac{M_s}{\varrho})^{\tau_{s-1}-1}\Big|\Psi(\varrho,x(\varrho),I^{\tau_{s-1}}_{M^+_{s-1}}x(\varrho))-\Psi(\varrho,y(\varrho),I^{\tau_{s-1}}_{M^+_{s-1}}y(\varrho))\Big|\frac{d\varrho}{\varrho}\\
&+\frac{|b|}{\Gamma(\tau_n)}\int_{M_n}^{M}(\log\frac{M}{\varrho})^{\tau_n-1}\Big|\Psi(\varrho,x(\varrho),I^{\tau_n}_{M^+_n}x(\varrho))-\Psi(\varrho,y(\varrho),I^{\tau_n}_{M^+_n}y(\varrho))\Big|\frac{d\varrho}{\varrho}\\
&+|b|\sum_{s=1}^{n}\Big|\Phi_s(x(M_s^-))-\Phi_s(y(M_s^-))\Big|\Big]\\
&+\sum_{0<M_s<t}\frac{1}{\Gamma(\tau_{s-1})}\int_{M_{s-1}}^{M_s}(\log\frac{M_s}{\varrho})^{\tau_{s-1}-1}\Big|\Psi(\varrho,x(\varrho),I^{\tau_{s-1}}_{M^+_{s-1}}x(\varrho))-\Psi(\varrho,y(\varrho),I^{\tau_{s-1}}_{M^+_{s-1}}y(\varrho))\Big|\frac{d\varrho}{\varrho}\\
&+\frac{1}{\Gamma(\tau_\vartheta)}\int_{M_\vartheta}^{t}(\log\frac{t}{\varrho})^{\tau_\vartheta-1}\Big|\Psi(\varrho,x(\varrho),I^{\tau_\vartheta}_{M^+_\vartheta}x(\varrho))-\Psi(\varrho,y(\varrho),I^{\tau_\vartheta}_{M^+_\vartheta}y(\varrho))\Big|\frac{d\varrho}{\varrho}\\
&+\sum_{0<M_s<t}\Big|\Phi_s(x(M_s^-))-\Phi_s(y(M_s^-))\Big|\\
\le\;& \frac{1}{|a+b|}\Big[|b|\sum_{s=1}^{n}\frac{1}{\Gamma(\tau_{s-1})}\int_{M_{s-1}}^{M_s}(\log\frac{M_s}{\varrho})^{\tau_{s-1}-1}\Big(D_1|x(\varrho)-y(\varrho)|+D_2|I^{\tau_{s-1}}_{M^+_{s-1}}(x(\varrho)-y(\varrho))|\Big)\frac{d\varrho}{\varrho}\\
&+\frac{|b|}{\Gamma(\tau_n)}\int_{M_n}^{M}(\log\frac{M}{\varrho})^{\tau_n-1}\Big(D_1|x(\varrho)-y(\varrho)|+D_2|I^{\tau_n}_{M^+_n}(x(\varrho)-y(\varrho))|\Big)\frac{d\varrho}{\varrho}\\
&+|b|\sum_{s=1}^{n}D_4\Big|x(M_s^-)-y(M_s^-)\Big|\Big]\\
&+\sum_{s=1}^{n}\frac{1}{\Gamma(\tau_{s-1})}\int_{M_{s-1}}^{M_s}(\log\frac{M_s}{\varrho})^{\tau_{s-1}-1}\Big(D_1|x(\varrho)-y(\varrho)|+D_2|I^{\tau_{s-1}}_{M^+_{s-1}}(x(\varrho)-y(\varrho))|\Big)\frac{d\varrho}{\varrho}\\
&+\frac{1}{\Gamma(\tau_\vartheta)}\int_{M_\vartheta}^{t}(\log\frac{t}{\varrho})^{\tau_\vartheta-1}\Big(D_1|x(\varrho)-y(\varrho)|+D_2|I^{\tau_\vartheta}_{M^+_\vartheta}(x(\varrho)-y(\varrho))|\Big)\frac{d\varrho}{\varrho}\\
&+\sum_{s=1}^{n}D_4\Big|x(M_s^-)-y(M_s^-)\Big|\\
\le\;& \frac{1}{|a+b|}\Big[|b|\frac{n(\log M)^{\tau_{s-1}}}{\Gamma(\tau_{s-1}+1)}\Big(D_1+D_2\frac{(\log M)^{\tau_{s-1}}}{\Gamma(\tau_{s-1}+1)}\Big)\|x-y\|\\
&+\frac{|b|(\log M)^{\tau_n}}{\Gamma(\tau_n+1)}\Big(D_1+D_2\frac{(\log M)^{\tau_n}}{\Gamma(\tau_n+1)}\Big)\|x-y\|\Big]+\frac{n(\log M)^{\tau_{s-1}}}{\Gamma(\tau_{s-1}+1)}\Big(D_1+D_2\frac{(\log M)^{\tau_{s-1}}}{\Gamma(\tau_{s-1}+1)}\Big)\|x-y\|\\
&+\frac{(\log M)^{\tau_\vartheta}}{\Gamma(\tau_\vartheta+1)}\Big(D_1+D_2\frac{(\log M)^{\tau_\vartheta}}{\Gamma(\tau_\vartheta+1)}\Big)\|x-y\|+\Big(\frac{|b|}{|a+b|}+1\Big)\sum_{s=1}^{n}D_4\|x-y\|\\
\le\;& \Big(\frac{|b|}{|a+b|}+1\Big)\Big[\frac{(n+1)(\log M)^{\tau^*}}{2(\sqrt{2}-1)}\Big(D_1+D_2\frac{(\log M)^{\tau^*}}{2(\sqrt{2}-1)}\Big)+nD_4\Big]\|x-y\|.
\end{aligned}
$$

Ergo, by (17), the operator S forms a contraction. Thus, S involves a fixed point uniquely which is the unique solution of the BVP (1)–(3). □

Remark 3. *Theorems 3 and 4 offer existence and uniqueness results for impulsive systems of fractional differential equations of the hybrid Caputo–Hadamard type with variable order. These* *results extend the results for differential equations with variable Caputo–Hadamard type fractional derivatives [27,28] to the impulsive case. Additionally, our results extend and complement some recently published results on impulsive fractional differential equations involving Caputo–Hadamard type constant order derivatives [41,42] considering variable order fractional differential equations.*

4. Ulam–Hyers Stability

To apply the obtained existence and uniqueness results, in this section we will consider the Ulam–Hyers stability of solutions of the BVP (1)–(3).

Theorem 5. *Consider the hypotheses of Theorem 4. Then, the BVP (1)–(3) is (**UH**) stable.*

Proof. Assume $z(t)$ satisfies the inequality (8). Then the integral inequality

$$\begin{aligned}
&z(t)+\frac{1}{a+b}\Big[b\sum_{s=1}^{n}\frac{1}{\Gamma(\tau_{s-1})}\int_{M_{s-1}}^{M_s}(\log\frac{M_s}{\varrho})^{\tau_{s-1}-1}\Psi(\varrho,z(\varrho),I_{M_{s-1}^+}^{\tau_{s-1}}z(\varrho))\frac{d\varrho}{\varrho}\\
&+\frac{b}{\Gamma(\tau_n)}\int_{M_n}^{M}(\log\frac{M}{\varrho})^{\tau_n-1}\Psi(\varrho,z(\varrho),I_{M_n^+}^{\tau_n}z(\varrho))\frac{d\varrho}{\varrho}+b\sum_{s=1}^{n}\Phi_s(z(M_s^-))-c\Big]\\
&-\sum_{0<M_s<t}\frac{1}{\Gamma(\tau_{s-1})}\int_{M_{s-1}}^{M_s}(\log\frac{M_s}{\varrho})^{\tau_{s-1}-1}\Psi(\varrho,z(\varrho),I_{M_{s-1}^+}^{\tau_{s-1}}z(\varrho))\frac{d\varrho}{\varrho}\\
&-\frac{1}{\Gamma(\tau_\vartheta)}\int_{M_\vartheta}^{t}(\log\frac{t}{\varrho})^{\tau_\vartheta-1}\Psi(\varrho,z(\varrho),I_{M_\vartheta^+}^{\tau_\vartheta}z(\varrho))\frac{d\varrho}{\varrho}-\sum_{0<M_s<t}\Phi_s(z(M_s^-))\le\epsilon\frac{M^{\tau^*}}{2(\sqrt{2}-1)}
\end{aligned}$$

holds.

According to Proposition 5, for $t\in(M_\vartheta,M_{\vartheta+1}],\ \vartheta=1,\dots,n$ the unique solution x of the BVP (1)–(3) is given by

$$\begin{aligned}
x(t)\ =\ &\frac{-1}{a+b}\Big[b\sum_{s=1}^{n}\frac{1}{\Gamma(\tau_{s-1})}\int_{M_{s-1}}^{M_s}(\log\frac{M_s}{\varrho})^{\tau_{s-1}-1}\Psi(\varrho,x(\varrho),I_{M_{s-1}^+}^{\tau_{s-1}}x(\varrho))\frac{d\varrho}{\varrho}\\
&+\frac{b}{\Gamma(\tau_n)}\int_{M_n}^{M}(\log\frac{M}{\varrho})^{\tau_n-1}\Psi(\varrho,x(\varrho),I_{M_n^+}^{\tau_n}x(\varrho))\frac{d\varrho}{\varrho}+b\sum_{s=1}^{n}\Phi_s(x(M_s^-))-c\Big]\\
&+\sum_{0<M_s<t}\frac{1}{\Gamma(\tau_{s-1})}\int_{M_{s-1}}^{M_s}(log\frac{M_s}{\varrho})^{\tau_{s-1}-1}\Psi(\varrho,x(\varrho),I_{M_{s-1}^+}^{\tau_{s-1}}x(\varrho))\frac{d\varrho}{\varrho}\\
&+\frac{1}{\Gamma(\tau_\vartheta)}\int_{M_\vartheta}^{t}(\log\frac{t}{\varrho})^{\tau_\vartheta-1}\Psi(\varrho,x(\varrho),I_{M_\vartheta^+}^{\tau_\vartheta}x(\varrho))\frac{d\varrho}{\varrho}+\sum_{0<M_s<t}\Phi_s(x(M_s^-)).
\end{aligned}$$

Let $t\in(M_\vartheta,M_{\vartheta+1}],\ \vartheta=1,\dots,n$. Then, we get

$$|z(t)-x(t)|$$

$$
\begin{aligned}
= \quad & \Big| z(t) + \frac{1}{a+b}\Big[b\sum_{s=1}^{n}\frac{1}{\Gamma(\tau_{s-1})}\int_{M_{s-1}}^{M_s}(\log\frac{M_s}{\varrho})^{\tau_{s-1}-1}\Psi(\varrho, x(\varrho), I^{\tau_{s-1}}_{M^+_{s-1}}x(\varrho))\frac{d\varrho}{\varrho} \\
& + \frac{b}{\Gamma(\tau_n)}\int_{M_n}^{M}(\log\frac{M}{\varrho})^{\tau_n-1}\Psi(\varrho, x(\varrho), I^{\tau_n}_{M^+_n}x(\varrho))\frac{d\varrho}{\varrho} + b\sum_{s=1}^{n}\Phi_s(x(M_s^-)) - c\Big] \\
& - \sum_{0<M_s<t}\frac{1}{\Gamma(\tau_{s-1})}\int_{M_{s-1}}^{M_s}(\log\frac{M_s}{\varrho})^{\tau_{s-1}-1}\Psi(\varrho, x(\varrho), I^{\tau_{s-1}}_{M^+_{s-1}}x(\varrho))\frac{d\varrho}{\varrho} \\
& - \frac{1}{\Gamma(\tau_\vartheta)}\int_{M_\vartheta}^{t}(\log\frac{t}{\varrho})^{\tau_\vartheta-1}\Psi(\varrho, x(\varrho), I^{\tau_\vartheta}_{M^+_\vartheta}x(\varrho))\frac{d\varrho}{\varrho} - \sum_{0<M_s<t}\Phi_s(x(M_s^-))\Big| \\
\leq \quad & \Big| z(t) + \frac{1}{a+b}\Big[b\sum_{s=1}^{n}\frac{1}{\Gamma(\tau_{s-1})}\int_{M_{s-1}}^{M_s}(\log\frac{M_s}{\varrho})^{\tau_{s-1}-1}\Psi(\varrho, z(\varrho), I^{\tau_{s-1}}_{M^+_{s-1}}z(\varrho))\frac{d\varrho}{\varrho} \\
& + \frac{b}{\Gamma(\tau_n)}\int_{M_n}^{M}(\log\frac{M}{\varrho})^{\tau_n-1}\Psi(\varrho, z(\varrho), I^{\tau_n}_{M^+_n}z(\varrho))\frac{d\varrho}{\varrho} + b\sum_{s=1}^{n}\Phi_s(z(M_s^-)) - c\Big] \\
& - \sum_{0<M_s<t}\frac{1}{\Gamma(\tau_{s-1})}\int_{M_{s-1}}^{M_s}(\log\frac{M_s}{\varrho})^{\tau_{s-1}-1}\Psi(\varrho, z(\varrho), I^{\tau_{s-1}}_{M^+_{s-1}}z(\varrho))\frac{d\varrho}{\varrho} \\
& - \frac{1}{\Gamma(\tau_\vartheta)}\int_{M_\vartheta}^{t}(\log\frac{t}{\varrho})^{\tau_\vartheta-1}\Psi(\varrho, z(\varrho), I^{\tau_\vartheta}_{M^+_\vartheta}z(\varrho))\frac{d\varrho}{\varrho} - \sum_{0<M_s<t}\Phi_s(z(M_s^-))\Big| \\
+ \quad & \frac{1}{|a+b|}\Big[|b|\sum_{s=1}^{n}\frac{1}{\Gamma(\tau_{s-1})}\int_{M_{s-1}}^{M_s}(\log\frac{M_s}{\varrho})^{\tau_{s-1}-1}\Big|\Psi(\varrho, z(\varrho), I^{\tau_{s-1}}_{M^+_{s-1}}z(\varrho)) - \Psi(\varrho, x(\varrho), I^{\tau_{s-1}}_{M^+_{s-1}}x(\varrho))\Big|\frac{d\varrho}{\varrho} \\
& + \frac{|b|}{\Gamma(\tau_n)}\int_{M_n}^{M}(\log\frac{M}{\varrho})^{\tau_n-1}\Big|\Psi(\varrho, z(\varrho), I^{\tau_n}_{M^+_n}z(\varrho)) - \Psi(\varrho, x(\varrho), I^{\tau_n}_{M^+_n}x(\varrho))\Big|\frac{d\varrho}{\varrho} \\
& + |b|\sum_{s=1}^{n}\Big|\Phi_s(z(M_s^-)) - \Phi_s(x(M_s^-))\Big|\Big] \\
& + \sum_{0<M_s<t}\frac{1}{\Gamma(\tau_{s-1})}\int_{M_{s-1}}^{M_s}(\log\frac{M_s}{\varrho})^{\tau_{s-1}-1}\Big|\Psi(\varrho, z(\varrho), I^{\tau_{s-1}}_{M^+_{s-1}}z(\varrho)) - \Psi(\varrho, x(\varrho), I^{\tau_{s-1}}_{M^+_{s-1}}x(\varrho))\Big|\frac{d\varrho}{\varrho} \\
& + \frac{1}{\Gamma(\tau_\vartheta)}\int_{M_\vartheta}^{t}(\log\frac{t}{\varrho})^{\tau_\vartheta-1}\Big|\Psi(\varrho, z(\varrho), I^{\tau_\vartheta}_{M^+_\vartheta}z(\varrho))\frac{d\varrho}{\varrho} - \Psi(\varrho, x(\varrho), I^{\tau_\vartheta}_{M^+_\vartheta}x(\varrho))\frac{d\varrho}{\varrho}\Big| \\
& + \sum_{0<M_s<t}\Big|\Phi_s(z(M_s^-)) - \Phi_s(x(M_s^-))\Big| \\
\leq \quad & \epsilon\frac{M^{\tau^*}}{2(\sqrt{2}-1)} \\
& + \frac{1}{|a+b|}\Big[|b|\sum_{s=1}^{n}\frac{1}{\Gamma(\tau_{s-1})}\int_{M_{s-1}}^{M_s}(\log\frac{M_s}{\varrho})^{\tau_{s-1}-1}\Big(D_1|z(\varrho)-x(\varrho)| + D_2|I^{\tau_{s-1}}_{M^+_{s-1}}(z(\varrho)-x(\varrho))|\Big)\frac{d\varrho}{\varrho} \\
& + \frac{|b|}{\Gamma(\tau_n)}\int_{M_n}^{M}(\log\frac{M}{\varrho})^{\tau_n-1}\Big(D_1|z(\varrho)-x(\varrho)| + D_2|I^{\tau_n}_{M^+_n}(z(\varrho)-x(\varrho))|\Big)\frac{d\varrho}{\varrho} \\
& + |b|\sum_{s=1}^{n}D_4\Big|z(M_s^-) - x(M_s^-)\Big|\Big] \\
& + \sum_{s=1}^{n}\frac{1}{\Gamma(\tau_{s-1})}\int_{M_{s-1}}^{M_s}(\log\frac{M_s}{\varrho})^{\tau_{s-1}-1}\Big(D_1|z(\varrho)-x(\varrho)| + D_2|I^{\tau_{s-1}}_{M^+_{s-1}}(z(\varrho)-x(\varrho))|\Big)\frac{d\varrho}{\varrho} \\
& + \frac{1}{\Gamma(\tau_\vartheta)}\int_{M_\vartheta}^{t}(\log\frac{t}{\varrho})^{\tau_\vartheta-1}\Big(D_1|z(\varrho)-x(\varrho)| + D_2|I^{\tau_\vartheta}_{M^+_\vartheta}(z(\varrho)-x(\varrho))|\Big)\frac{d\varrho}{\varrho} + \sum_{s=1}^{n}D_4\Big|z(M_s^-) - x(M_s^-)\Big| \\
\leq \quad & \epsilon\frac{M^{\tau^*}}{2(\sqrt{2}-1)} + \frac{1}{|a+b|}\Big[|b|\frac{n(\log M)^{\tau_{s-1}}}{\Gamma(\tau_{s-1}+1)}\Big(D_1 + D_2\frac{(\log M)^{\tau_{s-1}}}{\Gamma(\tau_{s-1}+1)}\Big)\|z-x\|
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{|b|(\log M)^{\tau_n}}{\Gamma(\tau_n+1)}\Big(D_1+D_2\frac{(\log M)^{\tau_n}}{\Gamma(\tau_n+1)}\Big)\|z-x\|\Big]+\frac{n(\log M)^{\tau_{s-1}}}{\Gamma(\tau_{s-1}+1)}\Big(D_1+D_2\frac{(\log M)^{\tau_{s-1}}}{\Gamma(\tau_{s-1}+1)}\Big)\|z-x\| \\
&+\frac{(\log M)^{\tau_\vartheta}}{\Gamma(\tau_\vartheta+1)}\Big(D_1+D_2\frac{(\log M)^{\tau_\vartheta}}{\Gamma(\tau_\vartheta+1)}\Big)\|z-x\|+\Big(\frac{|b|}{|a+b|}+1\Big)\sum_{s=1}^{n}D_4\|z-x\| \\
\leq\ &\epsilon\frac{M^{\tau^*}}{2(\sqrt{2}-1)}+\Big(\frac{|b|}{|a+b|}+1\Big)\Big[\frac{(n+1)(\log M)^{\tau^*}}{2(\sqrt{2}-1)}\Big(D_1+D_2\frac{(\log M)^{\tau^*}}{2(\sqrt{2}-1)}\Big)+nD_4\Big]\|z-x\|.
\end{aligned}
$$

Then,

$$
\|z-x\|\Big[1-\Big(\frac{|b|}{|a+b|}+1\Big)\Big[\frac{(n+1)(\log M)^{\tau^*}}{2(\sqrt{2}-1)}\Big(D_1+D_2\frac{(\log M)^{\tau^*}}{2(\sqrt{2}-1)}\Big)+nD_4\Big]\Big]\leq\epsilon\frac{M^{\tau^*}}{2(\sqrt{2}-1)}.
$$

Thus, we obtain,

$$
|z(t)-x(t)|\leq\frac{M^{\tau^*}}{2(\sqrt{2}-1)\Big[1-\Big(\frac{|b|}{|a+b|}+1\Big)\Big[\frac{(n+1)(\log M)^{\tau^*}}{2(\sqrt{2}-1)}\Big(D_1+D_2\frac{(logM)^{\tau^*}}{2(\sqrt{2}-1)}\Big)+nD_4\Big]\Big]}\epsilon:=c_\Psi\epsilon.
$$

Consequently, the BVP (1)–(3) is (**UH**) stable. □

Remark 4. *In [42], the authors studied the Ulam–Hyers stability of a class of Hadamard fractional differential equations with integral boundary value condition and impulses. Our Ulam–Hyers stability result extend and generalized these results to the case of variable order of fractional derivatives considering the hybrid type of Caputo–Hadamard derivatives. In fact, fractional derivatives of variable order are more general and expand the possibilities for applications of fractional-order models. Additionally, the stability result presented in this section show the applicability of the existence and uniqueness criteria established in Theorems 3 and 4 in the investigation of the qualitative properties of the introduced BVP (1)–(3).*

5. An Example

As an example, consider the following impulsive BVP,

$$
{}^{c}D^{\tau(t)}_{M_\vartheta^+}x(t)=\frac{e^{-3(t-1)}}{(\log t)^{\frac{1}{3}}(e^{e^{\frac{3(t-1)^2}{t}}}+5)(1+|x(t)|+|I^{\tau(t)}_{M_\vartheta^+}x(t)|)},\quad t\in\varpi:=\varpi_0\cup\varpi_1, \tag{18}
$$

$$
\Delta x|_{t=\frac{3}{2}}=\frac{|(x(\frac{3}{2}^-))|}{10+|(x(\frac{3}{2}^-))|}, \tag{19}
$$

$$
3x(1)+2x(M)=5, \tag{20}
$$

where

$$
M_0=1,\quad M_1=\frac{3}{2},\quad M_2=M=2,\quad n=1,\quad \varpi:=[1,2],\quad \varpi_0=[1,\frac{3}{2}],\quad \varpi_1=[\frac{3}{2},2],
$$

and

$$
\tau(t)=\begin{cases}\frac{1}{2}, & t\in\varpi_0,\\ \frac{3}{4}, & t\in\varpi_1.\end{cases} \tag{21}
$$

Let

$$
\Psi(t,x,y)=\frac{e^{-3(t-1)}}{(\log t)^{\frac{1}{3}}(e^{e^{\frac{3(t-1)^2}{t}}}+5)(1+|x|+|y|)},\ (t,x,y)\in\varpi\times\mathbb{R}\times\mathbb{R}.
$$

For each $t \in \varpi$ and $x_1, x_2, y_1, y_2 \in \mathbb{R}$, we have

$$(\log t)^{\frac{1}{3}}|\Psi(t, x_1, y_1) - \Psi(t, x_2, y_2)| \leq \frac{1}{e+5}(|x_1 - x_2| + |y_1 - y_2|).$$

Thus, assumption (A2) is satisfied with $D_1 = D_2 = \frac{1}{e+5}$ and $\gamma = \frac{1}{3}$.

Let

$$\Phi_1(x) = \frac{|x|}{7+|x|}, \quad x \in \mathbb{R}.$$

For all $x, y \in [0, \infty)$, we have

$$|\Phi_1(x) - \Phi_1(y) \leq \frac{1}{7}|x - y|.$$

Then, the assumption (A4) holds with $D_4 = \frac{1}{7}$.

We will also check that assumption (17) is fulfilled with $M = 2$, $n = 1$, $\gamma = \frac{1}{3}$, $D_1 = D_2 = \frac{1}{e+5}$, $D_4 = \frac{1}{7}$ and $\tau^* = \frac{3}{4}$. Indeed,

$$\Big(\frac{|b|}{|a+b|} + 1\Big)\Big[\frac{(n+1)(\log M)^{\tau^*}}{2(\sqrt{2}-1)}\Big(D_1 + D_2\frac{(\log M)^{\tau^*}}{2(\sqrt{2}-1)}\Big) + nD_4\Big] \simeq 0.8372 < 1.$$

Hence, assumption (17) is satisfied. By Theorem 4, the BVP (18)–(20) has a unique solution on $PC(\varpi, \mathbb{R})$.

In addition, according to Theorem 5, the BVP (18)–(20) is (**UH**) stable.

Remark 5. *The elaborated example again demonstrates the efficiency of our existence and uniqueness results. Additionally, it shows that the obtained fundamental criteria for differential equations with Caputo-type Hadamard derivatives of variable order and impulses can be easily applied in the study of their the Ulam–Hyers stability properties. Since the proposed criteria are represented as algebraic inequalities, they can be easily applied in the investigation of other qualitative properties of such equations.*

6. Conclusions

In this paper, we introduce a BVP for impulsive differential equations with Caputo–Hadamard fractional derivatives of variable order. We study the existence and uniqueness of solutions of the proposed fractional BVP. The obtained new results extend and complement some existing results on Caputo–Hadamard differential equations with constant-order fractional derivatives. The proposed existence and uniqueness criteria are also applied to establish Ulam–Hyers stability results. One example is presented to show the validity and applicability of the obtained results. The fundamental results presented in this paper open up many possibilities for future research. The obtained results can be applied in the qualitative study of the introduced fractional-order systems, such as stability, periodicity, almost periodicity, oscillations, asymptotic behavior, etc. In addition, it is possible to extend the proposed results to the uncertain case and study robust stability of such systems with uncertain terms. An important future topic is to apply the derived Ulam–Hyers stability results to fractional neural networks with Caputo–Hadamard fractional derivatives of variable order. An interesting future direction of research is to extend and implement the developed results to the class of fractional-order octonion-valued bidirectional associative memory neural networks introduced in [47] considering impulsive perturbations and variable order Caputo–Hadamard fractional derivatives. In addition, an analysis on the global Mittag–Leffler stability and synchronization problems for the established class of impulsive fractional differential equations and related neural network systems can be provided.

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