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Formulation of Impulsive Ecological Systems Using the Conformable Calculus Approach: Qualitative Analysis

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Abstract: In this paper, an impulsive conformable fractional Lotka–Volterra model with dispersion is introduced. Since the concept of conformable derivatives avoids some limitations of the classical fractional-order derivatives, it is more suitable for applied problems. The impulsive control approach which is common for population dynamics' models is applied and fixed moments impulsive perturbations are considered. The combined concept of practical stability with respect to manifolds is adapted to the introduced model. Sufficient conditions for boundedness and generalized practical stability of the solutions are obtained by using an analogue of the Lyapunov function method. The uncertain case is also studied. Examples are given to demonstrate the effectiveness of the established results.

Keywords: Lotka–Volterra system; conformable derivative; impulses; practical stability; manifolds

MSC: 26A33; 34A08; 34A37; 34D35

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1. Introduction

The mathematical modeling and the analysis of mathematical models are crucial objectives in the Mathematical Biology area. In fact, the establishing of mathematical models is very important in order to study the properties of various biological phenomena. Numerous researchers contributed to the development of innovative approaches to model and study the qualitative behavior of different classes of problems in biology and ecology. Among them, the Lotka–Volterra competitive systems are one of the most studied phenomena since they generalize several predator-prey models [1–3]. Different classes of Lotka–Volterra models has been proposed and investigated in the existing literature, including systems with dispersion [4–7].

It is worth indicating that all the above considered literature only involves integer-order Lotka–Volterra models. The development of new innovative modeling approaches is always a challenging task, and in their attempts to create more adequate models, recently the researchers considered fractional order dynamics. Due to the universality and flexibility provided and the wide range of applications in science, medicine and technologies, a great progress in fractional-order modeling has been made [8–11]. Fractional-order modeling approach has been applied to some important systems in biology and neuroscience [12–14]. The advantage of fractional-order models justified recent research activities in fractional-order formulation of Lotka–Volterra models [15–20]. Concerning fractional-order Lotka–Volterra dispersal models, the existing results are very seldom [21].

Another beneficial line of research is considering the effects of some short-term perturbations during the evolution of biological and ecological models. Impulsive differential equations are widely used to design and study the state behavior in such models [22–24], including impulsive fractional-order differential equations [25]. Also, due to the advantages of the impulsive control strategies [26], they have been proposed to mathematical models in biology. Indeed, impulsive effects are common in modeling biological and ecological processes since very often their behavior can be impulsively changed in response to some environmental fluctuations. Since random births and deaths of species may affect the qualitative behavior of a population dynamics systems, it is highly important to study the impulsive generalizations of the Lotka–Volterra models in which the states of the models are changed abruptly. Many authors answered the question of how short-term (impulsive) perturbations can be used to create impulsive control strategies for the qualitative properties of such systems [27–33]. There are also several existing results on impulsive fractional-order Lotka–Volterra models, although their number is very small [25,34]

The most popular definitions for fractional derivatives are the Caputo, Riemann–Liouville and Grunwald–Letnikov types [8,10,11]. However, the application of these derivatives to the qualitative analysis of fractional-order models is related to some limitations due to the absence of a simple chain rule formula, locality and singularity properties. These limitations motivate the researchers to introduce new definitions and avoid the restrictions of the existing ones [35–38].

The recently defined limit-based conformable or fractional-like derivative offers some computational simplifications related to derivatives of compositions of functions [39,40]. For some basic results on the fundamental and qualitative theory of conformable differential equations we refer to [41–46]. The fact that the use of such derivatives overcomes some difficulties in evaluating fractional derivatives motivates the research interest to apply them as innovative mathematical tools in modeling of real-world continuous systems [47–51].

In addition, some recent papers concern applications of conformable calculus on impulsive systems. However, the results on impulsive conformable problems are very rare [52–54]. With the little work on the application of the impulsive conformable approach, we are interested in expanding its application to Lotka–Volterra models. In fact, the analysis of biological processes depends on the appropriate choice of an fractional derivative. In this regard, it seems that the impulsive conformable modelling technique will be very suitable to be applied for models studied in population dynamics.

Stability and boundedness are two main problems in the qualitative study of mathematical models. Researchers constantly extended the classical stability theory to determine which movement mechanisms can support stability strategies that are acceptable from the practical point of view where the classical strategies do not allow a mathematically ideal stable behavior. One of the most used stability concept in this regard, is that of practical stability [55–58]. The recent results on practical stability of different classes of models have proven its remarkable importance [59–61].

The modification of the practical stability notion considering manifolds' practical stability instead of that of single solutions is even more powerful [24,25]. It is applied to some integer-order impulsive systems [62,63]. For impulsive conformable systems the concept has been studied in the papers [53,64]. Given the long history and vast literature on the practical stability with respect to manifolds notion, the problem of applying it to conformable Lotka–Volterra models with dispersion deserves our attention and this is one of the main goals of our study.

In this paper, we apply the impulsive conformable approach in modelling of Lotka-Volterra systems. We adapt the practical stability concept accompanied by the stability with respect to manifold notion as an extended combined stability strategy to study the behavior of the introduced model.

The novelty of the paper lies within the following few points:

1. A novel impulsive conformable Lotka–Volterra type model is introduced. Dispersion effects are also considered. The intraspecific coefficients are not neglected.

2. The modelling approach is a combination of the use of conformable derivatives with impulsive control perspective.

3. The concept of practical stability with respect to manifolds is adapted for studying the boundedness and stability manner of the introduced model.

4. The established conditions are new and include estimations of the model's parameters and impulsive control functions. As such they are uncomplicated for applications.

5. Finally, two examples are provided to demonstrate the correctness of the proposed results.

Through the text we will use the following basic notations: $\mathbb{R}_+ = [0, \infty)$, $\mathbb{R}$ denotes the set of all real numbers, $\mathbb{R}^n$ is the n-dimensional real space with the Euclidean norm $||x||$ of an $x \in \mathbb{R}^n$. For a continuous function $k(t)$ which is defined on J, $J \subseteq \mathbb{R}$, we denote

$$k^L = \inf_{t \in J} k(t), \quad k^M = \sup_{t \in J} k(t).$$

2. Problem Establishment and Preliminaries

2.1. Conformable Calculus

We will start this section with some definitions related to the conformable calculus from [39,40,44,53,54]. Let $t_0 \in \mathbb{R}_+$. Consider the points $t_1, t_2, \dots$,

$$t_0 < t_1 < t_2 \dots < t_k < t_{k+1} < \dots, \lim_{k \to \infty} t_k = \infty,$$

which will be considered as impulsive control instances for our model.

Definition 1. *[53,54] For any $\bar{t} \geq t_0$, $\mathcal{D}^q_{\bar{t}} u(t)$ is the generalized conformable derivative of order q, $0 < q \leq 1$ with the lower limit $\bar{t}$ for a function $u(t) : [\bar{t}, \infty) \to \mathbb{R}^n$ and is defined as*

$$\mathcal{D}^q_{\bar{t}}(u(t)) = \lim \left\{ \frac{u(t + \theta(t - \bar{t})^{1-q}) - u(t)}{\theta}, \; \theta \to 0 \right\}.$$

For $\bar{t} = t_k$, $k = 1, 2, \dots$, we have

$$\mathcal{D}^q_{t_k} u(t_k) = \lim_{t \to t_k^+} \mathcal{D}^q_{t_k} u(t).$$

Remark 1. *Since t_0 will not be considered as an impulsive point, for $\bar{t} = t_0$ Definition 1 is reduced to*

$$\mathcal{D}^q_{t_0} u(t) = \lim \left\{ \frac{u(t + \theta(t - t_0)^{1-q}) - u(t)}{\theta}, \; \theta \to 0 \right\},$$

which is applied in [39,40,44].

The class of all functions that have q-generalized conformable derivatives for any $t \in (\bar{t}, \infty)$ is denoted by $C^q((\bar{t}, b), \mathbb{R}^n)$. Such functions are known [53] as q-generalized conformable differentiable on $(\bar{t}, \infty)$.

Definition 2. *[53] The generalized conformable integral of order $0 < q \leq 1$ with a lower limit $\bar{t}$, $\bar{t} \geq t_0$, of a function $u : [\bar{t}, \infty) \to \mathbb{R}^n$ is defined as*

$$I^q_{\bar{t}} u(t) = \int_{\bar{t}}^{t} (\sigma - \bar{t})^{q-1} u(\sigma) d\sigma.$$

Throughout this paper, we will use the following properties of the generalized conformable derivatives [53,54].

Lemma 1. *[53] Let $x(u(t)) : (\bar{t}, \infty) \to \mathbb{R}$ and $0 < q \le 1$. If $x(\cdot)$ is differentiable with respect to $u(t)$ and $u(t)$ is q-generalized conformable differentiable on $(\bar{t}, \infty)$, then for any $t \in [\bar{t}, \infty)$ and $u(t) \neq 0$, we have*

$$\mathcal{D}_{\bar{t}}^{q} x(u(t)) = x'(u(t)) \mathcal{D}_{\bar{t}}^{q}(u(t)),$$

where x' is the derivative of $x(\cdot)$.

Remark 2. *Lemma 1 demonstrates the reasonableness of the use of the conformable fractional approach. Note that a similar result related to a simple application of the chain rule does not exist for the classical fractional-order derivatives.*

Lemma 2. *Let the function $u(t) : (\bar{t}, \infty) \to \mathbb{R}$ be q-generalized conformable differentiable on $(\bar{t}, \infty)$ for $0 < q \le 1$. Then for all $t > \bar{t}$*

$$I_{\bar{t}}^{q}(\mathcal{D}_{\bar{t}}^{q} u(t)) = u(t) - u(\bar{t}).$$

2.2. Model Formulation

In this paper we introduce a non-autonomous n-dimensional impulsive conformable Lotka–Volterra competitive system with dispersion and fixed moments of impulsive perturbations

$$\begin{cases} \mathcal{D}_{t_k}^{q} u_i(t) = u_i(t)\left[r_i(t) - a_{ii}(t)u_i(t) - \sum\limits_{j=1, j\neq i}^{n} a_{ij}(t)u_j(t)\right] \\ \qquad\qquad + \sum\limits_{j=1}^{n} b_{ij}(t)\big(u_j(t) - u_i(t)\big),\ t \neq t_k,\ k = 0, 1, \ldots, \\ \Delta u_i(t_k) = d_{ik} u_i(t_k),\ k = 1, 2, \ldots, \end{cases} \tag{1}$$

where $i = 1, 2, \ldots, n$, $n \ge 2$, $u_i(t)$ represents the population density of the i-th species at time t, the functions $r_i, a_{ii}, a_{ij}, b_{ij} \in C[\mathbb{R}, \mathbb{R}]$, $i, j = 1, 2, \ldots, n$ are the system's parameters, r_i denotes the intrinsic growth rate of the i-th species at time t, a_{ii} are the intraspecific coefficients, a_{ij} represent interspecific coefficients for species such that $i \neq j$, b_{ij} are dispersion rates, $\Delta u_i(t_k) = u_i(t_k^+) - u_i(t_k^-)$, the quantities $u_i(t_k)$ and $u_i(t_k^+)$ are, respectively, the population densities of species i before and after an impulsive jump at the moment t_k and the constants $d_{ik} \in \mathbb{R}$ represent the affect of the impulse perturbation on the species i at the moments t_k.

Remark 3. *The above model generalizes numerous existing integer-order Lotka–Volterra models [1–6], impulsive Lotka–Volterra models [27,29,30,32], as well as, Lotka–Volterra models with classical fractional-order derivatives [16–20,34] to the impulsive conformable case. In fact, the use of generalized conformable derivatives is motivated by their advantages in applications related to the simplifications in the use of the chain rule. Also, different from some existing conformable models [53], the effect of dispersion on the species which is an important subject in ecological models, and in population biology, more generally, is considered.*

Let $u_0 \in \mathbb{R}^n$. The solution of the model (1) which satisfies an initial condition of the type

$$u(t_0) = u_0 \tag{2}$$

will be denoted by $u(t) = u(t; t_0, u_0)$, where $u(t) = (u_1(t), u_2(t), \ldots, u_n(t))$.

At the moments t_k the following relations are satisfied:

$$u_i(t_k^-) = u_i(t_k),\ u_i(t_k^+) = u_i(t_k) + d_{ik} u_i(t_k). \tag{3}$$

It follows from (3) that the functions $u(t)$ that describe the states of the generalized conformable model (1) for different initial data are piecewise continuous functions with points of discontinuity of the first kind at which they are left continuous [53,54]. All such functions that are $q-$ generalized conformable differentiable on $\mathbb{R}_+$ form the space $PC^q(\mathbb{R}_+, \mathbb{R}^n)$.

In order to demonstrate the solutions of an impulsive conformable model, we will consider the next impulsive scalar generalized conformable equation

$$\begin{cases} \mathcal{D}^q_{t_k} x(t) = -\zeta x(t) + o(t),\ t \neq t_k,\ k = 0, 1, \ldots, \\ \\ \Delta x(t_k) = d_k x(t_k),\ k = 1, 2, \ldots. \end{cases} \tag{4}$$

where $x \in \mathbb{R}$, $\zeta > 0$, $o \in C(\mathbb{R}, \mathbb{R}_+)$, $d_k \in \mathbb{R}$, $k = 1, 2, \ldots$. Then, after application of the Definition 1 and the properties of the generalized conformable derivatives, for $t > t_k$ we have

$$\begin{aligned} x(t) = x(t_0) \textstyle\prod_{j=1}^k (1 + d_j) E_q(-\zeta, t_j - t_{j-1}) E_q(-\zeta, t - t_k) \\ + \textstyle\int_{t_k}^t W^q(t - t_k, \sigma - t_k)(\sigma - t_k)^{q-1} o(\sigma) d\sigma \\ + \textstyle\sum_{j=1}^k \prod_{l=k-j+1}^k (1 + d_l) E_q(-\zeta, t_l - t_{l-1}) \int_{t_{k-j}}^{t_{k-j+1}} W^q(t - t_k, \sigma - t_{k-j})(\sigma - t_{k-j})^{q-1} o(\sigma) d\sigma, \end{aligned} \tag{5}$$

where $W^q(t - t_k, \sigma - t_k) = E_q(-\zeta, t - t_k) E_q(\zeta, \sigma - t_k)$ and $E_q(\nu, \sigma)$ is the conformable exponential function given as [46]

$$E_q(\nu, \sigma) = \exp\left(\nu \frac{\sigma^q}{q}\right),\ \nu \in \mathbb{R}, \sigma \in \mathbb{R}_+.$$

We will further assume that any solution $u(t; t_0, u_0)$ of the initial value problem (IVP) (1), (2) corresponding to the initial data $(t_0, u_0) \in int(\mathbb{R}_+ \times \mathbb{R}^n)$ exists on $[t_0, \infty)$, and $u(t; t_0, u_0) \in PC^q([t_0, \infty), \mathbb{R}^n)$, $t \geq t_0$.

A solution $u(t) = \mathrm{col}(u_1(t), u_2(t), \ldots, u_n(t))$ of the model (1) is said to be strictly positive, if for $i = 1, 2, \ldots, n$,

$$0 < \inf_{t \in \mathbb{R}} u_i(t) \leq \sup_{t \in \mathbb{R}} u_i(t) < \infty.$$

2.3. Practical Stability with Respect to Manifolds Technique

In our qualitative analysis, we will adopt the powerful practical stability with respect to manifolds strategy [53,62–64] to the formulated model (1). To this end, we will define a manifold by a specific function.

Let $H : [t_0, \infty) \times \mathbb{R}^n \to \mathbb{R}^m$, $H = (H_1, H_2, \ldots, H_m)$, $m \leq n$, be a continuous function. We will study the practical stability of the next $(n - m)$-dimensional manifold which we will call a H-manifold:

$$\mathcal{M}^H_t = \{u \in \mathbb{R}^n :\ H_1(t, u) = H_2(t, u) = \cdots = H_m(t, u) = 0,\ t \in [t_0, \infty)\}. \tag{6}$$

Consider, also

$$\mathcal{M}^H_t(\varepsilon) = \{u \in \mathbb{R}^n :\ ||H(t, u)|| < \varepsilon,\ t \in [t_0, \infty)\},\ \varepsilon > 0,$$

and adopt the following notions [53,62–64].

Definition 3. *The manifold $\mathcal{M}^H_t$ is said to be:*

(a) practically stable for the model (1), if given (λ, A) with $0 < \lambda < A$, we have $u_0 \in \mathcal{M}^H_{t_0}(\lambda)$ implies $u(t; t_0, u_0) \in \mathcal{M}^H_t(A)$, $t \geq t_0$ for some $t_0 \in \mathbb{R}_+$;

(b) uniformly practically stable for the model (1), if (a) holds for every $t_0 \in \mathbb{R}_+$;

(c) practically asymptotically stable for the model (1), if (a) holds and

$$\lim_{t \to \infty} ||H(t, u(t; t_0, x_0))|| = 0;$$

(d) practically exponentially stable for the model (1), if given $(\lambda,\ A)$ *with* $0 < \lambda < A$, *we have* $u_0 \in \mathcal{M}^H_{t_0}(\lambda)$ *implies*

$$u(t;t_0,u_0) \in \mathcal{M}^H_t(A + \mu||H(t_0,u_0)||E_q(-\zeta, t-t_0)),\ t \geq t_0,\ \text{for some } t_0 \in \mathbb{R}_+,$$

where $0 < q < 1$, $\mu, \zeta > 0$.

Remark 4. *It is seen from Definition 3, that the extended practical stability with respect to a manifold concept generalizes several essential practical stability notions. Hence, it is a powerful technique which combine the benefits of the practical stability strategy with the stability with respect to manifolds techniques. In the case when* $H(t,u) = 0$ *only for* $u = 0$ *(for example if* $H(t,u) = u$*), then Definition 3 is reduced to the practical stability of the zero solution of the model (1). Similar case is the case, when* $H(t,u) = 0$ *only for* $u = u^*$, *where* u^* *is any state of interest to the model (1), such as equilibrium state, periodic or almost periodic state. The arbitrariness of the function* H *admits the consideration of several other particular cases of Definition 3. Also, since the practical stability concept allows the study of the states that are not mathematically ideally stable, it is more suitable for applied models that have manifolds (not single solutions) as asymptotic attractors.*

2.4. Conformable Lyapunov Functions Method

Consider the sets $\mathcal{G}_k = (t_{k-1}, t_k) \times \mathbb{R}^n$, $k = 1, 2, \ldots$, $\mathcal{G} = \bigcup_{k=1}^{\infty} \mathcal{G}_k$ and $Con_r = \{u \in \mathbb{R}^n : 0 < u_i < r\}$, $i = 1, 2, \ldots, n$, $r > 0$.

In the further considerations, we apply a modified conformable Lyapunov function approach. A class of Lyapunov-like functions $\mathcal{L}^q_k$ is defined as [53,62–64] a manifold of functions $L(t,u) : \mathcal{G} \to \mathbb{R}_+$ that are continuous on $\mathcal{G}$, q-generalized conformable differentiable in t, locally Lipschitz continuous with respect to u on each of the sets $\mathcal{G}_k$, $L(t,0) = 0$ for $t \geq t_0$, and for each $k = 1, 2, \ldots$ and $u \in \mathbb{R}^n$, there exist the finite limits

$$L(t_k^-, u) = \lim_{\substack{t \to t_k \\ t < t_k}} L(t,u), \quad L(t_k^+, u) = \lim_{\substack{t \to t_k \\ t > t_k}} L(t,u),$$

with $L(t_k^-, u) = L(t_k, u)$.

For a function $L \in \mathcal{L}^q_{t_k}$, $t > t_k$, the next is its upper right conformable derivative [53]

$$^+\mathcal{D}^q_{t_k}\, L(t,u) = \limsup \left\{ \frac{L(t + \theta(t-t_k)^{1-q},\ u(t + \theta(t-t_k)^{1-q}; t, x)) - L(t,u)}{\theta},\ \theta \to 0^+ \right\}. \tag{7}$$

Let for simplicity $f(t,u) = (f_1(t,u), f_2(t,u), \ldots, f_n(t,u))$, be

$$f_i(t,u) = u_i(t)\left[r_i(t) - \sum_{j=1}^{n} a_{ij}(t)u_j(t)\right] + \sum_{j=1}^{n} b_{ij}(t)\big(u_j(t) - u_i(t)\big)$$

for $i = 1, 2, \ldots, n$.

Then, the generalized conformable derivative of the function $L(t,u)$ with respect to the solution $u(t;t_k,u_k)$ of the problem (1) and (2) is given by [53]

$$^+\mathcal{D}^q_{t_k}\, L(t,u) = \limsup \left\{ \frac{L(t + \theta(t-t_k)^{1-q},\ u + \theta(t-t_k)^{1-q} f(t,u)) - L(t,u)}{\theta},\ \theta \to 0^+ \right\}. \tag{8}$$

For more information about the conformable modifications of the Lyapunov functions approach for impulsive control models, see [53,62–64].

We will also need the next Lemma from [53].

Lemma 3. *If for the function* $L \in \mathcal{L}^q_{t_k}$ *and for* $t \in [t_0, \infty)$, $u \in Con_r$, *we have:*

(i)

$$L(t_k^+, u) \leq L(t_k, u), \ \ k = 1, 2, \ldots,$$

(ii)

$$^{+}\mathcal{D}^{q}_{t_k} L(t,u) \leq -\zeta L(t,u) + o(t),\ t \neq t_k,\ k = 0,1,\ldots$$

for $\zeta = const > 0$, $o \in C^q(\mathbb{R}, \mathbb{R}_+)$, *then*

$$L(t,u(t)) \leq L(t_0^+, u_0)E_q(-\zeta, t-t_0) + \int_{t_k}^{t} \frac{W^q(t-t_k, \sigma - t_k)o(\sigma)}{(\sigma - t_k)^{1-q}} d\sigma$$

$$+ \sum_{j=1}^{k} \prod_{l=k-j+1}^{k} E_q(-\zeta, t_l - t_{l-1}) \int_{t_{k-j}}^{t_{k-j+1}} \frac{W^q(t-t_k, \sigma - t_{k-j})o(\sigma)}{(\sigma - t_{k_j})^{1-q}} d\sigma,\ t \geq t_0.$$

3. Comparison Results and Boundedness

In order to obtain practical stability results with respect to the manifold $\mathcal{M}_t^H$ related to system (1), firstly, we must prove some comparison results and boundedness criteria for all positive solutions of system (1).

We introduce the following basic hypotheses:

Hypothesis 1. *For the functions* $r_i(t)$, $a_{ii}(t)$, $1 \leq i \leq n$ *and* $a_{ij}(t)$, $b_{ij}(t)$, $1 \leq i,j \leq n$, $i \neq j$, $r_i^L > 0$, $r_i^M < \infty$, $a_{ii}^L > 0$, $a_{ii}^M < \infty$, $a_{ij}^L \geq 0$, $a_{ij}^M < \infty$, $b_{ij}^L \geq 0$, $b_{ij}^M < \infty$ *for* $1 \leq i,j \leq n$, $i \neq j$.

Hypothesis 2. *For any* $i = 1,2,\ldots,n$ *and* $k = 1,2,\ldots,$

$$1 + d_{ik} > 0.$$

Lemma 4. *If H1 and H2 hold, and* $u_i(t_0^+) > 0$, $i = 1,2,\ldots,n$, *then* $u_i(t) > 0$ *for* $i = 1,2,\ldots,n$.

Proof. Let us denote

$$p_i(t) = r_i(t) - \sum_{j=1}^{n} a_{ij}(t)u_j(t)$$

and

$$o_i(t) = \sum_{j=1}^{n} b_{ij}(t)\big(u_j(t) - u_i(t)\big).$$

Then, we can rewrite the system (1) in the form

$$\begin{cases} \mathcal{D}^q_{t_k} u_i(t) = u_i(t)p_i(t) + o_i(t),\ t \neq t_k,\ k = 0,1,\ldots, \\ \Delta u_i(t_k) = d_{ik}u_i(t_k),\ k = 1,2,\ldots, \end{cases} \tag{9}$$

$i = 1,2,\ldots,n$, and the proof follows from (5) and H1, H2. □

Lemma 5. *Assume that:*

1. *Hypotheses H1 and H2 hold and* $u_i(t_0^+) > 0$, $i = 1,2,\ldots,n$.
2. $\mathcal{U}(t) = \mathrm{col}(\mathcal{U}_1(t), \mathcal{U}_2(t), \ldots, \mathcal{U}_n(t))$ *is the maximal solution of*

$$\begin{cases} \mathcal{D}^q_{t_k} \mathcal{U}_i(t) = \mathcal{U}_i(t)\Big[r_i^M - a_{ii}^L \mathcal{U}_i(t)\Big] + \sum\limits_{j=1, j\neq i}^{n} b_{ij}^M \mathcal{U}_j(t),\ t \neq t_k,\ k = 0,1,\ldots, \\ \\ \Delta \mathcal{U}_i(t_k) = d_k^M \mathcal{U}_i(t_k),\ k = 1,2,\ldots, \end{cases}$$

where $d_k^M = \max\{d_{ik}\}$ *for* $i = 1,2,\ldots,n$ *and* $k = 1,2,\ldots.$

3. $\mathcal{W}(t) = \mathrm{col}(\mathcal{W}_1(t), \mathcal{W}_2(t), \dots, \mathcal{W}_n(t))$ *is the minimal solution of*

$$\begin{cases} \mathcal{D}^q_{t_k}\mathcal{W}_i(t) = \mathcal{W}_i(t)\Big[r_i^L - \sum\limits_{j=1, j\neq i}^n b_{ij}^M - a_{ii}^M \mathcal{W}_i(t) - \sum\limits_{\substack{j=1\\ j\neq i}}^n a_{ij}^M \sup\limits_{t\geq t_0} \mathcal{U}_j(t)\Big],\ t\neq t_k,\ k=0,1,\dots, \\ \Delta\mathcal{W}_i(t_k) = d_k^L \mathcal{W}_i(t_k),\ k=1,2,\dots, \end{cases}$$

where $d_k^L = \min\{d_{ik}\}$ *for* $i = 1,2,\dots,n$ *and* $k = 1,2,\dots$.

4. For each $i = 1,2,\dots,n$*,*

$$0 \leq \mathcal{W}_i(t_0^+) \leq u_i(t_0^+) \leq \mathcal{U}_i(t_0^+).$$

Then,

$$\mathcal{W}_i(t) \leq u_i(t) \leq \mathcal{U}_i(t),\ t\geq t_0,\ 1\leq i\leq n. \tag{10}$$

Proof. Let $t > t_k$. From Lemma 4 for the system (1) it follows

$$\begin{aligned} \mathcal{D}^q_{t_k} u_i(t) &= u_i(t)\Bigg[r_i(t) - a_{ii}(t)u_i(t) - \sum_{j=1, j\neq i}^n a_{ij}(t)u_j(t)\Bigg] + \sum_{j=1}^n b_{ij}(t)\big(u_j(t) - u_i(t)\big) \\ &\leq u_i(t)\Bigg[r_i(t) - a_{ii}(t)u_i(t) - \sum_{j=1, j\neq i}^n a_{ij}(t)u_j(t)\Bigg] + \sum_{j=1, j\neq i}^n b_{ij}(t)u_j(t) \\ &\leq u_i(t)\Bigg[r_i(t) - a_{ii}(t)u_i(t)\Bigg] + \sum_{j=1, j\neq i}^n b_{ij}(t)u_j(t) \end{aligned}$$

and then

$$\begin{cases} \mathcal{D}^q_{t_k} u_i(t) \leq u_i(t)\Big[r_i^M - a_{ii}^L u_i(t)\Big] + \sum\limits_{j=1, j\neq i}^n b_{ij}^M u_j(t),\ t\neq t_k,\ k=0,1,\dots, \\ \Delta u_i(t_k) \leq d_k^M u_i(t_k),\ k=1,2,\dots. \end{cases}$$

By analogy, we get

$$\begin{cases} \mathcal{D}^q_{t_k} u_i(t) \geq u_i(t)\Big[r_i^L - \sum\limits_{j=1, j\neq i}^n b_{ij}^M - a_{ii}^M u_i(t) - \sum\limits_{\substack{j=1\\ j\neq i}}^n a_{ij}^M \sup\limits_{t\geq t_0} u_j(t)\Big],\ t\neq t_k,\ k=0,1,\dots, \\ \Delta u_i(t_k) \geq d_k^L u_i(t_k),\ k=1,2,\dots. \end{cases}$$

Then, using Lemma 4, the properties of the generalized conformable derivatives, and Conditions 2, 3 and 4 of Lemma 5 we obtain (1) for $t \geq t_0$ and $1 \leq i \leq n$. □

Lemma 6. *If, in addition to the conditions of Lemma 5, we have*

$$r_i^L \geq \sum_{j=1, j\neq i}^n \frac{1}{a_j^L} a_{ij}^M \Big(r_j^M + \sum_{j=1, j\neq i}^n b_{ij}^M\Big) + \sum_{j=1, j\neq i}^n b_{ij}^M,\ i=1,2,\dots,n,$$

then,

$$\begin{aligned} &\min\Big\{u_i(t_0^+), \frac{1}{a_i^M}\Big[r_i^L - \sum_{j=1, j\neq i}^n b_{ij}^M - \sum_{j=1, j\neq i}^n \frac{1}{a_j^L} a_{ij}^M \Big(r_j^M + \sum_{j=1, j\neq i}^n b_{ij}^M\Big)\Big]\Big\} \\ &\leq u_i(t) \leq \max\Big\{u_i(t_0^+), \frac{1}{a_i^L}\Big(r_j^M + \sum_{j=1, j\neq i}^n b_{ij}^M\Big)\Big\},\ i=1,2,\dots,n \end{aligned}$$

for any $t \in [t_0, t_1] \cup (t_k, t_{k+1}]$, $k = 1,2,\dots$.

Proof. From Lemma 5 it follows that the inequalities (9) hold for $t \geq t_0 \in \mathbb{R}_+$ and $i = 1, 2, \ldots, n$.

We will prove that

$$\frac{1}{a_i^M}\Big[r_i^L + \sum_{j=1, j\neq i}^{n} b_{ij}^L - \sum_{j=1, j\neq i}^{n} \frac{1}{a_j^L} a_{ij}^M \Big(r_j^M + \sum_{j=1, j\neq i}^{n} b_{ij}^M\Big)\Big] \leq \mathcal{W}_i(t)$$

$$\leq \mathcal{U}_i(t) \leq \frac{1}{a_i^L}\Big(r_j^M + \sum_{j=1, j\neq i}^{n} b_{ij}^M\Big),$$

for all $t \in [t_0, t_1] \cup (t_k, t_{k+1}]$, $k = 1, 2, \ldots$ and $i = 1, 2, \ldots, n$.

First, we will prove the inequality

$$\mathcal{U}_i(t) \leq \frac{1}{a_i^L}\Big(r_j^M + \sum_{j=1, j\neq i}^{n} b_{ij}^M\Big),\ t \in [t_0, t_1] \cup (t_k, t_{k+1}],\ k = 1, 2, \ldots,\ i = 1, 2, \ldots, n. \tag{11}$$

If we suppose that exists a $t \in [t_0, t_1] \cup (t_k, t_{k+1}]$, $k = 1, 2, \ldots$ such that for some i with $i = 1, 2, \ldots, n$, we have

$$\mathcal{U}_i(t) > \frac{1}{a_i^L}\Big(r_j^M + \sum_{j=1, j\neq i}^{n} b_{ij}^M\Big),$$

then using Definition 1 and Lemma 4, we will get

$$\mathcal{D}_{t_k}^{q}\mathcal{U}_i(t) < \mathcal{U}_i(t)\Big[r_i^M - a_{ii}^L \mathcal{U}_i(t)\Big] + \sup_t \mathcal{U}_j(t) \sum_{j=1, j\neq i}^{n} b_{ij}^M < 0,$$

which is a contradiction. Hence, the inequality (11) holds for all i, $1 \leq i \leq n$ and all $t \in [t_0, t_1] \cup (t_k, t_{k+1}]$, $k = 1, 2, \ldots$.

The proof of the inequality

$$\mathcal{W}_i(t) \geq \frac{1}{a_i^M}\Big[r_i^M - \sum_{j=1, j\neq i}^{n} b_{ij}^L - \sum_{j=1, j\neq i}^{n} \frac{1}{a_j^L} a_{ij}^M \Big(r_j^M + \sum_{j=1, j\neq i}^{n} b_{ij}^M\Big)\Big]$$

for i, $i = 1, 2, \ldots, n$ and all $t \in [t_0, t_1] \cup (t_k, t_{k+1}]$, $k = 1, 2, \ldots$ is similar. □

Next, the following boundedness result will be presented.

Theorem 1. *Under the conditions of Lemma 6, if*

$$1 + d_{ik} \leq 1, \tag{12}$$

then, there exist constants $0 \leq \alpha_i < \beta_i$*, such that*

$$\alpha_i \leq u_i(t) \leq \beta_i,\ t \geq t_0,\ i = 1, 2, \ldots, n.$$

Proof. Lemma 6 implies the existence of positive constants α_i^* and β_i^* such that

$$\alpha_i^* \leq u_i(t) \leq \beta_i^*$$

for all i, $i = 1, 2, \ldots, n$ and all $t \in [t_0, t_1] \cup (t_k, t_{k+1}]$, $k = 1, 2, \ldots$.

From (12), we have

$$0 < u_i(t_k^+) = (1 + d_{ik}) u_i(t_k) \leq u_i(t_k) \leq \beta_i^*.$$

The above inequalities lead to the assertion of Theorem 1. □

4. Practical Stability with Respect to Manifolds Results

The aim of this paper is the establishment of practical stability criteria for the manifold $\mathcal{M}_t^H$. First, we will present a practical exponential stability result.

We will assume that for the continuous function H the sets $\mathcal{M}_t^H$, $\mathcal{M}_t^H(\varepsilon)$ are $(n-m)$-dimensional manifolds in $\mathbb{R}^n$, and each solution $u(t;t_0,u_0)$ of the model (1) with an initial condition (2) which satisfies

$$||H(t,u(t;t_0,u_0))|| \leq \Omega < \infty$$

is defined for $t \geq t_0$.

Theorem 2. *For given $0 < \lambda < A$, assume that conditions of Theorem 1 hold, there exists a function $L \in \mathcal{L}_{t_k}^q$ such that for $t \in [t_0,\infty)$, $u \in Con_r$,*

$$||H(t,u)|| \leq L(t,u) \leq \Theta(\Omega)||H(t,u)||,\ \Theta(\Omega) \geq 1,\ 0 < \Omega < \infty, \tag{13}$$

and for the model's parameter we have

$$0 < \zeta^* < \frac{\alpha(1+\alpha)}{1+\beta}\sum_{j=1}^n a_{ji}(t),\ t \geq t_0,\ \alpha = \min_i \alpha_i,\ \beta = \max_i \beta_i,\ i = 1,2,\ldots,n, \tag{14}$$

$$\begin{aligned}\mathcal{G}(t) = &\int_{t_0}^{\infty} \frac{W^q(t-t_k,\sigma-t_k)}{(\sigma-t_0)^{1-q}}\left(\sum_{i=1}^n \left(r_i(\sigma) + \frac{\beta-\alpha}{1+\alpha}\sum_{j=1}^n b_{ij}(\sigma)\right)\right)d\sigma \\ &+ \sum_{j=1}^k \prod_{l=k-j+1}^k E_q(-\zeta^*, t_l - t_{l-1}) \\ &\times \int_{t_{k-j}}^{t_{k-j+1}} \frac{W^q(t-t_k,\sigma-t_{k-j})}{(\sigma-t_{k_j})^{1-q}}\left(\sum_{i=1}^n \left(r_i(\sigma) + \frac{\beta-\alpha}{1+\alpha}\sum_{j=1}^n b_{ij}(\sigma)\right)\right)d\sigma < \infty,\end{aligned} \tag{15}$$

then the manifold $\mathcal{M}_t^H$ is practically exponentially stable for the model (1).

Proof. Let $0 < \lambda < A$. No generality is lost by making the assumption $1 < \lambda < G(t) < A$.

Consider the following Lyapunov-like function

$$L(u) = \sum_{i=1}^n \ln(1+u_i).$$

From H2 and (12), we have that for $t_k > t_0 \geq 0$, $k = 1,2,\ldots,$

$$L(u(t_k^+)) = \sum_{i=1}^n \ln(1+u_i(t_k^+)) = \sum_{i=1}^n \ln[1+(1+d_{ik})u_i(t_k)] \leq L(u(t_k)). \tag{16}$$

From Theorem 1, for $t \neq t_k$, $k = 1,2,\ldots,$ we have

$$\begin{aligned}{}^+\mathcal{D}_{t_k}^q\, L(u(t)) &\leq \sum_{i=1}^n \frac{1}{1+u_i(t)}\mathcal{D}_{t_k}^q u_i(t) \\ &\leq \sum_{i=1}^n \frac{u_i(t)}{1+u_i(t)}r_i(t) - \frac{\alpha}{1+\beta}\sum_{i=1}^n\sum_{j=1}^n a_{ij}(t)u_j(t) + \sum_{i=1}^n\sum_{j=1}^n \frac{1}{1+u_i(t)}b_{ij}(t)\big(u_j(t)-u_i(t)\big) \\ &\leq -\zeta^*\sum_{j=1}^n \ln(1+u_j(t)) + \sum_{i=1}^n r_i(t) + \frac{\beta-\alpha}{1+\alpha}\sum_{j=1}^n b_{ij}(t)\end{aligned}$$

$$= -\zeta^* L(u(t)) + \Big(\sum_{i=1}^{n} r_i(t) + \frac{\beta-\alpha}{\alpha}\sum_{j=1}^{n} b_{ij}(t)\Big).$$

The last inequality and Lemma 3 imply

$$L(u(t)) \leq L(u(t_0^+))E_q(-\zeta^*, t-t_0) + \mathcal{G}(t),\ t \geq t_0. \tag{17}$$

If $u_0 \in \mathcal{M}_{t_0}^H(\lambda)$, then from the choice of the function $L \in \mathcal{L}_{t_k}^q$, (13) and (17), we have

$$||H(t, u(t; t_0, u_0))|| \leq L(u(t; t_0, u_0))$$

$$< A + \Theta(\Omega)||H(t_0, u_0)||E_q(-\zeta^*, t-t_0), \quad t \geq t_0.$$

Therefore,

$$u(t; t_0, u_0) \in \mathcal{M}_t^H(A + \Theta(\Omega)||H(t_0, u_0)||E_q(-\zeta^*, t-t_0),\ t \geq t_0, \text{ for some } t_0 \in \mathbb{R}_+,$$

which means that the manifold $\mathcal{M}_t^H$ is practically exponentially stable for the model (1). □

Finally, in order to study the effect of uncertain terms and robust practical stability behavior of the model (1), we consider the following system

$$\begin{cases} \mathcal{D}_{t_k}^q u_i(t) = u_i(t)\Big[r_i(t) + \tilde{r}_i(t) - \sum_{j=1}^{n}\big(a_{ij}(t) + \tilde{a}_{ij}(t)\big)u_j(t)\Big] \\ \qquad + \sum_{j=1}^{n}\big(b_{ij}(t) + \tilde{b}_{ij}(t)\big)\big(u_j(t) - u_i(t)\big),\ t \neq t_k,\ k = 0, 1, \ldots, \\ u_i(t_k^+) = u_i(t_k) + d_{ik}u_i(t_k) + \tilde{d}_{ik}u_i(t_k),\ k = 1, 2, \ldots, \end{cases} \tag{18}$$

where the functions $\tilde{r}_i,\ \tilde{a}_{ji}, \tilde{b}_{ij} \in C[\mathbb{R}_+, \mathbb{R}_+],\ i, j = 1, \ldots, n,\ k = 1, 2, \ldots$ and constants $\tilde{d}_{ik},\ i = 1, 2, \ldots, n,\ k = 1, 2, \ldots,$ represent uncertainties in the parameters [65–67].

The consideration of the model (18) with uncertain parameters is motivated by the fact that the characteristics of an ecological model may be affected by changes in environment or some noises. Also, since the robust stability and control theories are main optimal control techniques, their development is important.

Definition 4. *The manifold $\mathcal{M}_t^H$ is called practically robustly exponentially stable for the model (1) if for $t_0 \in \mathbb{R}_+$, $u_0 \in \mathcal{M}_{t_0}^H(\lambda)$ and for any $\tilde{r}_i$, $\tilde{a}_{ji}$, $\tilde{b}_{ij}$, $\tilde{d}_{ik}$, $i, j = 1, \ldots, n$, $k = 1, 2, \ldots$, the manifold $\mathcal{M}_t^H$ is practically exponentially stable with respect to system (18).*

Using similar technique and steps as in the proof of Theorem 2, we can verify the next result.

Theorem 3. *Assume that conditions of Theorem 2 hold, $-1 < d_{ik} + \tilde{d}_{ik} \leq 0$, $i = 1, 2, \ldots, n$, $k = 1, 2, \ldots$, and for the model's parameter we have*

$$0 < \zeta^{**} < \frac{\alpha(1+\alpha)}{1+\beta}\sum_{j=1}^{n}(a_{ji}(t) + \tilde{a}_{ji}(t)),\ t \geq t_0,\ \alpha = \min_i \alpha_i,\ \beta = \max_i \beta_i,\ i = 1, 2, \ldots, n, \tag{19}$$

$$\mathcal{G}^*(t) = \int_{t_0}^{\infty} \frac{W^q(t-t_k, \sigma-t_k)}{(\sigma-t_0)^{1-q}} \sum_{i=1}^{n}\Big(r_i(\sigma) + \tilde{r}_i(\sigma) + \frac{\beta-\alpha}{1+\alpha}\sum_{j=1}^{n}\Big(b_{ij}(\sigma) + \tilde{b}_{ij}(\sigma)\Big)\Big)d\sigma$$

$$+ \sum_{j=1}^{k}\prod_{l=k-j+1}^{k} E_q(-\zeta^{**}, t_l - t_{l-1})$$

$$\times \int_{t_{k-j}}^{t_{k-j+1}} \frac{W^q(t-t_k,\sigma-t_{k-j})}{(\sigma-t_{k_j})^{1-q}} \sum_{i=1}^{n} \Big(r_i(\sigma) + \tilde{r}_i(\sigma) + \frac{\beta-\alpha}{1+\alpha} \sum_{j=1}^{n} \Big(b_{ij}(\sigma) + \tilde{b}_{ij}(\sigma) \Big) \Big) d\sigma < \infty,$$

then the manifold $\mathcal{M}_t^H$ is practically robustly exponentially stable for the model (1).

5. Illustrative Examples

In this Section, examples are addressed to illustrate the usefulness of the proposed method.

Example 1. *We consider the 2-dimensional impulsive dispersal Lotka–Volterra system with generalized conformable derivatives*

$$\begin{cases} \mathcal{D}_{t_k}^q u_1(t) = u_1(t)[r_1(t) - a_{11}(t)u_1(t) - a_{12}(t)u_2(t)] + b_{12}(t)\big(u_2(t) - u_1(t)\big),\ t \neq t_k,\ k = 0,1,\dots \\ \mathcal{D}_{t_k}^q u_2(t) = u_2(t)[r_2(t) - a_{21}(t)u_1(t) - a_{22}(t)u_2(t)] + b_{21}(t)\big(u_1(t) - u_2(t)\big),\ t \neq t_k,\ k = 0,1,\dots \\ u_1(t_k^+) = \dfrac{u_1(t_k)}{3},\ k = 1,2,\dots, \qquad u_2(t_k^+) = \dfrac{u_2(t_k)}{4},\ k = 1,2,\dots, \end{cases} \tag{20}$$

where $t_k < t_{k+1} < \dots$, $k = 1,2,\dots$, $\lim_{k\to\infty} t_k = \infty$, $r_1(t) = 5 - \sin t$, $r_2(t) = 6 - \sin t$, $a_{11}(t) = 0.02$, $a_{12}(t) = a_{21} = 0.01$, $a_{22}(t) = 0.05$, the the dispersal rates are $b_{12}(t) = 0.3$, $b_{21}(t) = 0.5$.

For the system (20), we have that $d_{1k} = -\frac{2}{3}$, $d_{2k} = -\frac{3}{4}$ and, hence conditions H2 and (12) are satisfied.

Also, $r_1^L = 4$, $r_1^M = 6$, $r_2^l = 5$, $r_2^M = 7$, $a_{11}^L = a_{11}^M = 0.02$, $a_{12}^L = a_{12}^M = a_{21}^L = a_{21}^M = 0.01$, $a_{22}^L = a_{22}^M = 0.05$, $b_{12}^L = b_{12}^M = 0.3$, $b_{21}^L = b_{21}^M = 0.5$, and H1 and conditions of Lemma 6 are satisfied.

Therefore, by Theorem 1, we can conclude that there exist constants $0 \le \alpha_i < \beta_i$, such that

$$\alpha_i \le u_i(t) \le \beta_i,\ t \ge t_0,\ i = 1,2.$$

Let us consider a solution $u^* = (u_1^*, u_2^*)$ of the model (20), the function $H : \mathbb{R}^2 \to \mathbb{R}^2$, $H = (H_1, H_2) = (u_1 - u_1^*, u_2 - u_2^*)$, and the manifold

$$\mathcal{M}_t^H = \{u \in \mathbb{R}^2 : H_1 = H_2 = 0\}.$$

We have that (13) is satisfied, and for the values of α, β and ζ^* for which (14) and (15) hold, the manifold $\mathcal{M}_t^H$ is practically exponentially stable for the model (20).

Example 2. *Keeping the parameters' values from the model (20) we consider the following system with uncertain parameters We consider the 2-dimensional impulsive dispersal Lotka–Volterra system with generalized conformable derivatives*

$$\begin{cases} \mathcal{D}_{t_k}^q u_1(t) = u_1(t)\Big[r_1(t) + \tilde{r}_1(t) - \big(a_{11}(t) + \tilde{a}_{11}(t)\big)u_1(t) - \big(a_{12}(t) + \tilde{a}_{12}(t)\big)u_2(t)\Big] \\ \qquad + \big(b_{12}(t) + \tilde{b}_{12}(t)\big)\big(u_2(t) - u_1(t)\big),\ t \neq t_k,\ k = 0,1,\dots, \\ \mathcal{D}_{t_k}^q u_2(t) = u_1(t)\Big[r_2(t) + \tilde{r}_2(t) - \big(a_{21}(t) + \tilde{a}_{21}(t)\big)u_1(t) - \big(a_{22}(t) + \tilde{a}_{22}(t)\big)u_2(t)\Big] \\ \qquad + \big(b_{21}(t) + \tilde{b}_{22}(t)\big)\big(u_1(t) - u_2(t)\big),\ t \neq t_k,\ k = 0,1,\dots, \\ \Delta u_1(t_k) = \Big(-\dfrac{2}{3} + \tilde{d}_{1k}\Big)u_1(t_k),\ k = 1,2,\dots,\ \ \Delta u_2(t_k) = \Big(-\dfrac{3}{4} + \tilde{d}_{ik}\Big)u_2(t_k),\ k = 1,2,\dots, \end{cases} \tag{21}$$

where $t_k < t_{k+1} < \dots$, $k = 1,2,\dots$, $\lim_{k\to\infty} t_k = \infty$, $\tilde{r}_i$, $\tilde{a}_{ji}, \tilde{b}_{ij} \in C[\mathbb{R}_+, \mathbb{R}_+]$, $i,j = 1,2$, $k = 1,2,\dots$ and $\tilde{d}_{ik}$ are uncertain constants, $i = 1,2$, $k = 1,2,\dots$.

In fact, the model (20) is the nominal model for the uncertain system (21). Also, if all uncertain values are bounded, and the conditions of Theorem 3 are met, this will guarantee the practical robust exponential stability of the manifold $\mathcal{M}_t^H$ *for the model (20).*

6. Conclusions

In this paper, we introduce an impulsive Lotka–Volterra-type models using the conformable calculus approach. The introduced model extends and complements numerous existing integer-order Lotka–Volterra models [1–6], impulsive Lotka–Volterra models [27,29,30,32], as well as, Lotka–Volterra models with classical fractional-order derivatives [16–20,34] to the impulsive conformable case. The benefits of the conformable derivatives make the introduced model more relevant to the real-world applications. The effect of dispersion on the species which is an important subject in ecological models, is also considered. The combined extended practical stability with respect to manifolds notion is adopted to the introduced model, and sufficient conditions are derived to ensure the boundedness and practical stability with respect to manifolds by using Lyapunov function. The uncertain case is also studied to contribute to the development of the robust stability and control theories. Two illustrative examples are given to demonstrate the effectiveness of the contributed results. The application of the proposed conformable fractional calculus approach to some neural network models is an interesting topic for a future research. In addition, it is possible to extend the proposed results to the delayd case and study the effect of some delay effects on the qualitative behavior of the states.

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