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Quantum Laplace Transforms for the Ulam–Hyers Stability of Certain q-Difference Equations of the Caputo-like Type

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Abstract: We aim to investigate the stability property for the certain linear and nonlinear fractional q-difference equations in the Ulam–Hyers and Ulam–Hyers–Rassias sense. To achieve this goal, we prove that three types of the linear q-difference equations of the q-Caputo-like type are Ulam–Hyers stable by using the quantum Laplace transform and quantum Mittag–Leffler function. Moreover, after proving the existence property for a nonlinear Cauchy q-difference initial value problem, we use the same quantum Laplace transform and the q-Gronwall inequality to show that it is generalized Ulam–Hyers–Rassias stable.

Keywords: q-derivative; quantum Laplace transform; q-Mittag–Leffler function; q-Gronwall inequality; Ulam–Hyers stability

MSC: 34A08; 34A60; 35A23; 39A60

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1. Introduction

Stability analysis is a very important branch of applied mathematics that deals with a wide range of linear and nonlinear differential and difference equations in continuous and discrete calculus, respectively. In fact, this field of mathematics analyzes the behavior of the solutions of dynamic systems when they suffer from very small disturbances in their input parameters.

Since, in real-world mathematical models, the exact solution of a dynamic system is crucial, we need to find these solutions based on a standard method, but, in most of the cases, finding the exact solutions is complicated or impossible. Therefore, researchers have turned to numerical or analytical techniques to obtain approximate solutions. However, what is important here is whether such approximate solutions are stable compared to exact solutions. For the first time, Ulam discussed the stability of the solution in a scientific lecture in 1940. After this, Ulam and Hyers extended their theory in the framework of several papers [1,2].

Due to the interesting results presented by these mathematicians, the theory of stability became a major topic among other mathematicians, and its effectiveness can be clearly seen in many scientific papers published in recent decades. Nowadays, this theory is known as the Ulam–Hyers and Ulam–Hyers–Rassias stability. Next, similar studies were conducted in this direction, and researchers presented new generalized versions of the stability notion in the Ulam–Hyers sense. In 2015, Xu et al. [3] used the fixed point technique to investigate the Ulam–Hyers stability for higher-order linear differential equations. In 2020, Liu et al. [4] conducted another stability analysis for solutions of a fractional differential equation equipped with a kernel of the Mittag–Leffler type. For more details, see [5–10].

In addition to analytical techniques, some integral transforms have also helped us to find explicit solutions for some differential equations with constant coefficients. Of course, these integral transforms have different forms; some of the most famous and widely used ones are the Laplace, Mellin, and Fourier transforms. Later, these transforms also showed their effectiveness in the topic of the stability of solutions. Kadiev and Ponosow [11] used a new method based on the W-transform to conduct an analysis on the stability of stochastic linear difference equations. In 2013, Rezaei et al. [12] applied the Laplace transform method to evaluate the Ulam–Hyers stability of the n-th-order differential equations in the linear form. Next, Rezaei, along with other authors, completed a study using the Fourier transform on $\mathcal{L}^1(\mathbb{R})$ [13]. In 2015, Wang and Xu [14] continued to conduct research on the Ulam–Hyers stability for two types of Caputo fractional differential equations via the Laplace transform method.

In 2019, Liu et al. [15] studied the Ulam–Hyers stability and its generalized type on a linear differential equation of the Caputo–Fabrizio type with the Laplace transform method. In 2020, Ramdoss et al. [16] conducted an Ulam–Hyers stability analysis for two homogeneous and non-homogeneous n-th-order differential equations with constant coefficients, given by

$$\mathtt{y}^{(\mathtt{n})}(\mathtt{t}) + \lambda_1 \mathtt{y}^{(\mathtt{n}-1)}(\mathtt{t}) + \cdots + \lambda_{\mathtt{n}-1}\mathtt{y}'(\mathtt{t}) + \lambda_{\mathtt{n}}\mathtt{y}(\mathtt{t}) = 0,$$

and

$$\mathtt{y}^{(\mathtt{n})}(\mathtt{t}) + \lambda_1 \mathtt{y}^{(\mathtt{n}-1)}(\mathtt{t}) + \cdots + \lambda_{\mathtt{n}-1}\mathtt{y}'(\mathtt{t}) + \lambda_{\mathtt{n}}\mathtt{y}(\mathtt{t}) = \mathtt{g}(\mathtt{t}),$$

respectively, by using the Fourier transform.

In 2021, Rezapour et al. [17] considered a nonlinear system of the coupled Caputo–Navier boundary value problems and investigated its stability and existence properties by using the generalized differential transform method (GDT-method). In the same year, Etemad et al. [18] conducted similar research for a system of coupled thermostat control boundary problems via the GDT method. In 2022, Ganesh et al. [19] combined the Fourier transform method with the Ulam–Hyers Mittag–Leffler stability for the solutions of two fractional differential equations. In 2023, Pinelas et al. [20] used a new transform, named the general integral transform, introduced by Jafari [21], and studied the Ulam–Hyers stability with the help of this technique.

In 1910, a new form of calculus was developed in which derivatives were defined without the limit notion. For the first time, Jackson [22,23] presented such derivatives and their corresponding integrals in the context of quantum calculus, or q-calculus. Later, researchers found some applications for these newly defined q-operators. Fock [24] considered an q-difference equation and established that all atoms of hydrogen have the symmetry property. Other researchers completed studies on the properties of q-operators and q-series, which developed some novel concepts in quantum mechanics, hypergeometric functions, combinatorics, and orthogonal polynomials [25,26]. Moreover, there are recent articles in relation to applications of q-operators in mathematical models and inequality theory, such as [27–31].

In 2014, Chung, Kim, and Kwon [32] introduced an q-analogue of the Laplace transform, named the quantum Laplace transform or q-Laplace transform, and proved some important properties of this type of integral transform. After presenting some needed lemmas in this regard, we try to combine this transform with the stability field on some special linear and nonlinear fractional q-difference equations of the Caputo-like type. More precisely, based on the existing properties of the quantum Laplace transform, and inspired by the above-mentioned papers, an Ulam–Hyers stability analysis is conducted on the following linear q-difference equations of the Caputo-like type, given as

$$({}^{c}\mathbb{D}^{\mathtt{r}}_{(\mathtt{q})}\mathtt{y})(\mathtt{t}) = \mathtt{h}(\mathtt{t}), \tag{1}$$

$$({}^{c}\mathbb{D}^{\mathtt{r}}_{(\mathtt{q})}\mathtt{y})(\mathtt{t}) - \alpha\mathtt{y}(\mathtt{t}) = \mathtt{h}(\mathtt{t}), \tag{2}$$

$$({}^{c}\mathbb{D}^{r_1}_{(q)}y)(t) - \alpha({}^{c}\mathbb{D}^{r_2}_{(q)}y)(t) = h(t), \tag{3}$$

where $q \in (0,1)$ is the quantum parameter, $r, r_1, r_2 \in (0,1)$ with $r_2 < r_1$ and $\alpha \in \mathbb{R}$. Also, ${}^{c}\mathbb{D}^{\beta}_{(q)}$ denotes the q-derivative of the Caputo-like type of the order $\beta \in \{r, r_1, r_2\}$, and $h : [0,\infty) \to \mathbb{R}$ is a continuous function.

In the following, the nonlinear q-type Cauchy difference equation with the initial value

$$\begin{cases} ({}^{c}\mathbb{D}^{r}_{(q)}y)(t) = \phi(t, y(t)), \\ y(0) = y_0, \quad 0 < r < 1. \end{cases} \tag{4}$$

is considered to study the existence of the solutions. To prove the Ulam–Hyers–Rassias stability, we use the quantum Laplace transform along with the q-Gronwall inequality with respect to the q-Mittag–Leffler function.

Note that instead of implementing our computations on specific models, we have tried to consider the general form of the fractional q-difference equations. Therefore, since the objectives of the study are fulfilled on these q-difference equations, this method can easily be performed on any mathematical model of difference equations caused by engineering and medical processes and achieve the desired results. Therefore, the main contributions of this paper can be considered as follows:

1. The use of quantum Laplace transforms for the first time to prove the stability of solutions;
2. The use of the q-Mittag–Leffler function to find the convergent series of the solutions;
3. The easier application of series in quantum computing and their exact outputs in computer programming;
4. The relation of the q-Gronwall inequality in the context of the stability of the fractional q-difference equations.

We arrange the structure of the paper as follows. Some preliminaries are recalled in Section 2 as a reminder. Section 3 deals with the Ulam–Hyers stability for three linear q-difference equations by using the quantum Laplace transform method. Section 4 continues the same technique for an q-type Cauchy problem of the q-Caputo-like type. After an illustrative example in Section 5, the conclusions are given in Section 6.

2. Preliminaries

In order to better understand the concepts and theorems in the following sections, we review some of the most important definitions and properties of the operators and transformations. In this paper, let $0 < q < 1$ be the quantum parameter.

2.1. Ulam–Hyers Stability

Inspired by [1,2], we present some definitions of Ulam–Hyers stable solutions as a reminder.

Definition 1 ([1])**.** *We say that the fractional* q*-difference equation*

$$F\big(h, y, {}^{c}\mathbb{D}^{r_1}_{(q)}y, \ldots, {}^{c}\mathbb{D}^{r_n}_{(q)}y\big) = 0, \tag{5}$$

admits the Ulam–Hyers stability if a real constant $K > 0$ *exists so that, for all* $\epsilon > 0$ *and each solution function* $y \in \mathcal{C}([0,T],\mathbb{R})$ *of* $F\big(h, y, {}^{c}\mathbb{D}^{r_1}_{(q)}y, \ldots, {}^{c}\mathbb{D}^{r_n}_{(q)}y\big) \le \epsilon$*, there is a solution function* $\tilde{y} \in \mathcal{C}([0,T],\mathbb{R})$ *of the given* q*-difference Equation (5) so that* $|y(t) - \tilde{y}(t)| \le K\epsilon$*.*

Definition 2 ([2])**.** *We say that the fractional* q*-difference equation*

$$F\big(h, y, {}^{c}\mathbb{D}^{r_1}_{(q)}y, \ldots, {}^{c}\mathbb{D}^{r_n}_{(q)}y\big) = 0, \tag{6}$$

admits the generalized Ulam–Hyers–Rassias stability with respect to the function $\Lambda \in \mathcal{C}([0, \mathtt{T}], \mathbb{R})$ *if a real constant* $K > 0$ *exists so that, for each solution function* $\mathtt{y} \in \mathcal{C}([0, \mathtt{T}], \mathbb{R})$ *of*

$$\mathtt{F}\big(\mathtt{h}, \mathtt{y}, {}^{\mathtt{c}}\mathbb{D}^{\mathtt{r}_1}_{(\mathtt{q})}\mathtt{y}, \ldots, {}^{\mathtt{c}}\mathbb{D}^{\mathtt{r}_n}_{(\mathtt{q})}\mathtt{y}\big) \leq \Lambda(\mathtt{t}),$$

there is a solution function $\tilde{\mathtt{y}} \in \mathcal{C}([0, \mathtt{T}], \mathbb{R})$ *of the given* q*-difference Equation* (6) *so that*

$$|\mathtt{y}(\mathtt{t}) - \tilde{\mathtt{y}}(\mathtt{t})| \leq K\Lambda(\mathtt{t}).$$

2.2. q-*Calculus*

The q-numbers (denoted by $[\cdot]_{(\mathtt{q})}$) play an important role in q-calculus. They are the q-analogues of ordinary numbers [33]. To define these extended numbers, let $\mathtt{r} \in \mathbb{R}$ be arbitrary. In this case,

$$[\mathtt{r}]_{(\mathtt{q})} = \mathtt{q}^{\mathtt{r}-1} + \cdots + \mathtt{q} + 1 = \frac{1 - \mathtt{q}^{\mathtt{r}}}{1 - \mathtt{q}}, \ (\mathtt{r} \neq 0).$$

Accordingly, for $\mathtt{n} \in \mathbb{N}$, we have $[\mathtt{n}]_{(\mathtt{q})}! = [\mathtt{n}]_{(\mathtt{q})}[\mathtt{n} - 1]_{(\mathtt{q})} \ldots [2]_{(\mathtt{q})}[1]_{(\mathtt{q})}$. This defines the notion of the q-factorial $[\cdot]_{(\mathtt{q})}!$. Note that $[0]_{(\mathtt{q})} = 0$.

Definition 3 ([33])**.** *Let* $\mathtt{n} \in \mathbb{N}$ *and* $\mathtt{r} \in \mathbb{R}$*. The* q*-shifted power functions are defined by*

$$(\mathtt{t} - \tau)^{(\mathtt{n})}_{(\mathtt{q})} = \prod_{\mathtt{j}=0}^{\mathtt{n}-1}(\mathtt{t} - \tau\mathtt{q}^{\mathtt{j}}) \quad \text{and} \quad (\mathtt{t} - \tau)^{(\mathtt{r})}_{(\mathtt{q})} = \mathtt{t}^{\mathtt{r}} \prod_{\mathtt{j}=0}^{\infty} \frac{\mathtt{t} - \tau\mathtt{q}^{\mathtt{j}}}{\mathtt{t} - \tau\mathtt{q}^{\mathtt{r}+\mathtt{j}}},$$

so that $0 \leq \tau \leq \mathtt{t}$ *and* $(\mathtt{t} - \tau)^{(0)}_{(\mathtt{q})} = 1$.

Note that $(\mathtt{t})^{(\mathtt{r})}_{(\mathtt{q})} = \mathtt{t}^{\mathtt{r}}$ if $\tau = 0$ [33].

Definition 4. *By* $\Gamma_{(\mathtt{q})}(\cdot)$*, we denote the* q*-Gamma function, which is given by*

$$\Gamma_{(\mathtt{q})}(\mathtt{r}) = (1 - \mathtt{q})^{1-\mathtt{r}}(1 - \mathtt{q})^{(\mathtt{r}-1)}, \quad \mathtt{r} \notin \mathbb{N} \cup \{0\}.$$

By the definition of the q-Gamma function, it is clear that $\Gamma_{(\mathtt{q})}(\mathtt{n}) = [\mathtt{n} - 1]_{(\mathtt{q})}!$, and, for each $\mathtt{r} \in \mathbb{R}$, we have $\Gamma_{(\mathtt{q})}(\mathtt{r} + 1) = [\mathtt{r}]_{(\mathtt{q})}\Gamma_{(\mathtt{q})}(\mathtt{r})$ [33].

Definition 5 ([34])**.** *The* q*-derivative (Jackson sense) for a real function* y *is defined by*

$$(\mathbb{D}_{(\mathtt{q})}\mathtt{y})(\mathtt{t}) = \frac{\mathtt{y}(\mathtt{t}) - \mathtt{y}(\mathtt{q}\mathtt{t})}{(1 - \mathtt{q})\mathtt{t}},$$

and the q*-integral (Jackson sense) is defined by*

$$\int_0^{\mathtt{t}} \mathtt{y}(\mathtt{w})\, \mathrm{d}_{(\mathtt{q})}\mathtt{w} = (1 - \mathtt{q}) \sum_{\mathtt{k}=0}^{\infty} \mathtt{t}\mathtt{q}^{\mathtt{k}}\mathtt{y}(\mathtt{t}\mathtt{q}^{\mathtt{k}}).$$

Definition 6 ([35,36])**.** *The fractional* q*-integral of the Riemann–Liouville-like type for the function* $\mathtt{y} \in \mathcal{C}([0, +\infty), \mathbb{R})$ *of order* $\mathtt{r} > 0$ *is defined as*

$$({}^{\mathtt{R}}\mathbb{I}^{\mathtt{r}}_{(\mathtt{q})}\mathtt{y})(\mathtt{t}) = \frac{1}{\Gamma_{(\mathtt{q})}(\mathtt{r})} \int_0^{\mathtt{t}} (\mathtt{t} - \mathtt{q}\mathtt{w})^{(\mathtt{r}-1)}_{(\mathtt{q})} \mathtt{y}(\mathtt{w})\, \mathrm{d}_{(\mathtt{q})}\mathtt{w},$$

and ${}^{\mathtt{R}}\mathbb{I}^{0}_{(\mathtt{q})}\mathtt{y}(\mathtt{t}) = \mathtt{y}(\mathtt{t})$. *Moreover, the fractional* q*-derivative of the Caputo-like type is defined as*

$$({}^{\mathtt{c}}\mathbb{D}^{\mathtt{r}}_{(\mathtt{q})}\mathtt{y})(\mathtt{t}) = \frac{1}{\Gamma_{(\mathtt{q})}(\mathtt{k}-\mathtt{r})}\int_0^{\mathtt{t}} (\mathtt{t}-\mathtt{q}\mathtt{w})^{(\mathtt{k}-\mathtt{r}-1)}_{(\mathtt{q})}(\mathbb{D}^{\mathtt{k}}_{(\mathtt{q})}\mathtt{y})(\mathtt{w})\, d_{(\mathtt{q})}\mathtt{w},$$

so that $\mathtt{k}-1 < \mathtt{r} \le \mathtt{k}$.

Definition 7 ([37]). *The* q*-exponential functions are defined as*

$$\mathrm{EXP}_{\mathtt{q}}(-\mathtt{t}) = \sum_{\mathtt{k}=0}^{\infty} \frac{(-1)^{\mathtt{k}}\mathtt{q}^{\binom{\mathtt{k}}{2}}}{[\mathtt{k}]_{(\mathtt{q})}!}\mathtt{t}^{\mathtt{k}}, \quad \textit{and} \quad \exp_{\mathtt{q}}(\mathtt{t}) = \sum_{\mathtt{k}=0}^{\infty}\frac{1}{[\mathtt{k}]_{(\mathtt{q})}!}\mathtt{t}^{\mathtt{k}}.$$

Definition 8 ([38]). *Let* $\alpha \in \mathbb{R}$ *and* $\mathtt{r}_1, \mathtt{r}_2 \in \mathbb{C}$ *so that* $\mathrm{Re}(\mathtt{r}_1) > 0$ *and* $\mathrm{Re}(\mathtt{r}_2) > 0$. *Then, the* q*-Mittag–Leffler function about* $\mathtt{t}_0$ *is defined as*

$$\mathbf{E}^{\mathtt{q}}_{\mathtt{r}_1,\mathtt{r}_2}(\alpha, \mathtt{t}-\mathtt{t}_0) = \sum_{\mathtt{k}=0}^{\infty}\frac{\alpha^{\mathtt{k}}(\mathtt{t}-\mathtt{t}_0)^{(\mathtt{r}_1\mathtt{k})}_{(\mathtt{q})}}{\Gamma_{(\mathtt{q})}(\mathtt{r}_2+\mathtt{r}_1\mathtt{k})}, \qquad \mathtt{t}, \mathtt{t}_0 \in \mathbb{C}.$$

In the special case, we define

$$\mathbf{E}^{\mathtt{q}}_{\mathtt{r}_1}(\alpha, \mathtt{t}-\mathtt{t}_0) = \sum_{\mathtt{k}=0}^{\infty}\frac{\alpha^{\mathtt{k}}(\mathtt{t}-\mathtt{t}_0)^{(\mathtt{r}_1\mathtt{k})}_{(\mathtt{q})}}{\Gamma_{(\mathtt{q})}(1+\mathtt{r}_1\mathtt{k})}, \qquad \mathtt{t}, \mathtt{t}_0 \in \mathbb{C}.$$

Now, we state the q-type of the Gronwall inequality. This inequality is useful in proving the stability theorems.

Theorem 1 ([39]). *Suppose that* $\mathtt{r} > 0$, $0 < \mathtt{q} < 1$ *and the functions* $\mathtt{y}(\mathtt{t})$, $\mathtt{x}(\mathtt{t})$ *and* $\mathtt{z}(\mathtt{t})$ *are non-negative so that* $\mathtt{x}(\mathtt{t})$ *and* $\mathtt{z}(\mathtt{t})$ *are also non-decreasing for* $\mathtt{t} \ge 0$. *Also, let* $\mathtt{z}(\mathtt{t}) \le C$ *so that* C *is a constant. If*

$$\mathtt{y}(\mathtt{t}) \le \mathtt{x}(\mathtt{t}) + \frac{\mathtt{z}(\mathtt{t})}{\Gamma_{(\mathtt{q})}(\mathtt{r})}\int_0^{\mathtt{t}}(\mathtt{t}-\mathtt{q}\mathtt{w})^{(\mathtt{r}-1)}_{(\mathtt{q})}\mathtt{y}(\mathtt{w})\, d_{(\mathtt{q})}\mathtt{w},$$

then

$$\mathtt{y}(\mathtt{t}) \le \mathtt{x}(\mathtt{t})\mathbf{E}^{\mathtt{q}}_{\mathtt{r}}(\alpha, \mathtt{z}(\mathtt{t})\Gamma_{(\mathtt{q})}(\mathtt{r})).$$

2.3. Quantum Laplace Transform

This subsection deals with the definition and some properties of the quantum Laplace transform.

Definition 9 ([32]). *The quantum Laplace transform (or* q*-Laplace transform), for a given function* $\mathtt{y}(\mathtt{t})$*, is defined by*

$$\mathbb{L}_{\mathtt{q}}\Big(\mathtt{y}(\mathtt{t})\Big)(\tau) := \mathtt{Y}(\tau) = \int_0^{\infty}\mathrm{EXP}_{\mathtt{q}}(-\mathtt{q}\tau\mathtt{t})\mathtt{y}(\mathtt{t})\, d_{(\mathtt{q})}\mathtt{t}, \quad (\tau > 0).$$

This transform has the linearity property, i.e.,

$$\mathbb{L}_{\mathtt{q}}\Big(c_1\mathtt{y}_1(\mathtt{t}) + c_2\mathtt{y}_2(\mathtt{t})\Big)(\tau) = c_1\mathbb{L}_{\mathtt{q}}\Big(\mathtt{y}_1(\mathtt{t})\Big)(\tau) + c_2\mathbb{L}_{\mathtt{q}}\Big(\mathtt{y}_2(\mathtt{t})\Big)(\tau), \quad (c_1, c_2 \in \mathbb{R}).$$

The quantum Laplace transforms of some functions are collected in the following lemma.

Lemma 1 ([32]). *Let* $0 < \mathtt{q} < 1$ *and* $\mathtt{r} \in \mathbb{R}$ *with* $\mathtt{r} > -1$. *Then,*

1. $\mathbb{L}_{\mathtt{q}}\Big(\mathtt{t}^{\mathtt{r}}\Big)(\tau) = \frac{1}{\tau^{\mathtt{r}+1}}\Gamma_{(\mathtt{q})}(\mathtt{r}+1)$*; particularly, we have* $\mathbb{L}_{\mathtt{q}}\Big(1\Big)(\tau) = \frac{1}{\tau}$.

2. $\mathbb{L}_q\Big(t^n\Big)(\tau) = \frac{[n]_{(q)}!}{\tau^{n+1}}, \quad (n \in \mathbb{N}).$
3. $\mathbb{L}_q\Big(\mathrm{EXP}_q(\lambda t)\Big)(\tau) = \sum_{k=0}^{\infty}(-1)^k \frac{q^{\binom{k}{2}}}{\tau^{k+1}}\lambda^k, \quad (\lambda \in \mathbb{R}).$

The following lemma presents the quantum Laplace transforms of the q-derivatives (Jackson sense) of higher orders.

Lemma 2 ([32])**.** *For a given continuous function $y(t)$, let the functions $(\mathbb{D}_{(q)}y)(t), \ldots, (\mathbb{D}^{n-1}_{(q)}y)(t)$ be also continuous on $(0,\infty)$. Moreover, let $(\mathbb{D}^n_{(q)}y)(t)$ be piecewise continuous on the same interval. Then,*

1. $\mathbb{L}_q\Big((\mathbb{D}_{(q)}y)(t)\Big)(\tau) = -y(0) + \tau\mathbb{L}_q\Big(y(t)\Big)(\tau).$
2. $\mathbb{L}_q\Big((\mathbb{D}^2_{(q)}y)(t)\Big)(\tau) = -\tau y(0) - (\mathbb{D}_{(q)}y)(0) + \tau^2\mathbb{L}_q\Big(y(t)\Big)(\tau).$
3. $\mathbb{L}_q\Big((\mathbb{D}^n_{(q)}y)(t)\Big)(\tau) = \tau^n\mathbb{L}_q\Big(y(t)\Big)(\tau) - \sum_{k=0}^{n-1}\tau^{n-1-k}(\mathbb{D}^k_{(q)}y)(0), \quad (n \in \mathbb{N}).$

The quantum Laplace transform of the fractional q-derivative of the Caputo-like type for a given function is stated below.

Lemma 3 ([40])**.** *Let $y(t)$ and $(\mathbb{D}_{(q)}y)(t)$ be continuous. Then,*

$$\mathbb{L}_q\Big(({}^c\mathbb{D}^r_{(q)}y)(t)\Big)(\tau) = -y(0)\tau^{r-1} + \tau^r\mathbb{L}_q\Big(y(t)\Big)(\tau), \quad 0 < r < 1.$$

Lemma 4 ([40])**.** *Let y_1 and y_2 be two functions. Then,*

$$\mathbb{L}_q\Big((y_1 *_q y_2)(t)\Big)(\tau) = \mathbb{L}_q\Big(y_1(t)\Big)(\tau) \cdot \mathbb{L}_q\Big(y_2(t)\Big)(\tau),$$

where the quantum convolution $y_1 *_q y_2$ for two functions y_1 and y_2 is defined as

$$(y_1 *_q y_2)(t) = \int_0^t y_1(w)y_2(t - qw)\, d_{(q)}w.$$

Now, we provide three propositions whose conclusions are needed for our main theorems. In fact, by the definition of the q-Mittag–Leffler functions (Definition 8), we prove three important equalities as follows.

Proposition 1. *Let $\alpha \in \mathbb{R}$ and $\mathrm{Re}(r) > 0$. Then,*

$$\mathbb{L}_q\Big(\mathbf{E}^q_r(\alpha, t)\Big)(\tau) = \frac{\tau^{r-1}}{\tau^r - \alpha}, \quad \Big|\frac{\alpha}{\tau^r}\Big| < 1.$$

Proof. We know that

$$\mathbf{E}^q_r(\alpha, t) = \sum_{k=0}^{\infty}\frac{\alpha^k (t)^{(rk)}_{(q)}}{\Gamma_{(q)}(1 + rk)}, \quad t \in \mathbb{C}. \tag{7}$$

Applying the quantum Laplace transform on (7) and by Lemma 1, we have

$$\begin{aligned}\mathbb{L}_q\Big(\mathbf{E}^q_r(\alpha, t)\Big)(\tau) &= \mathbb{L}_q\Big(\sum_{k=0}^{\infty}\frac{\alpha^k (t)^{(rk)}_{(q)}}{\Gamma_{(q)}(1 + rk)}\Big)(\tau) \\ &= \sum_{k=0}^{\infty}\frac{\alpha^k}{\Gamma_{(q)}(1 + rk)}\mathbb{L}_q\Big(t^{rk}\Big)(\tau)\end{aligned}$$

$$= \sum_{k=0}^{\infty} \frac{\alpha^k}{\Gamma_{(q)}(1+rk)} \frac{\Gamma_{(q)}(1+rk)}{\tau^{1+rk}}$$

$$= \sum_{k=0}^{\infty} \frac{\alpha^k}{\tau^{1+rk}}$$

$$= \frac{1}{\tau} \sum_{k=0}^{\infty} \left(\frac{\alpha}{\tau^r}\right)^k$$

$$= \frac{1}{\tau} \cdot \frac{\tau^r}{\tau^r - \alpha}$$

$$= \frac{\tau^{r-1}}{\tau^r - \alpha}.$$

The proof is completed. □

Proposition 2. *Let* $\alpha \in \mathbb{R}$ *and* $\mathrm{Re}(r) > 0$. *Then,*

$$\mathbb{L}_q\Big(t^{r-1}\mathbf{E}^q_{r,r}(\alpha,t)\Big)(\tau) = \frac{1}{\tau^r - \alpha}, \quad \Big|\frac{\alpha}{\tau^r}\Big| < 1.$$

Proof. We know that

$$\mathbf{E}^q_{r,r}(\alpha,t) = \sum_{k=0}^{\infty} \frac{\alpha^k (t)^{(rk)}_{(q)}}{\Gamma_{(q)}(r+rk)}, \quad t \in \mathbb{C}.$$

In this case, we have

$$t^{r-1}\mathbf{E}^q_{r,r}(\alpha,t) = t^{r-1} \sum_{k=0}^{\infty} \frac{\alpha^k (t)^{(rk)}_{(q)}}{\Gamma_{(q)}(r+rk)}, \quad t \in \mathbb{C}. \tag{8}$$

Applying the quantum Laplace transform on (8) and by Lemma 1, we have

$$\mathbb{L}_q\Big(t^{r-1}\mathbf{E}^q_{r,r}(\alpha,t)\Big)(\tau) = \mathbb{L}_q\Big(t^{r-1} \sum_{k=0}^{\infty} \frac{\alpha^k (t)^{(rk)}_{(q)}}{\Gamma_{(q)}(r+rk)}\Big)(\tau)$$

$$= \mathbb{L}_q\Big(\sum_{k=0}^{\infty} \frac{\alpha^k t^{rk+r-1}}{\Gamma_{(q)}(r+rk)}\Big)(\tau)$$

$$= \sum_{k=0}^{\infty} \frac{\alpha^k}{\Gamma_{(q)}(r+rk)} \mathbb{L}_q\Big(t^{rk+r-1}\Big)(\tau)$$

$$= \sum_{k=0}^{\infty} \frac{\alpha^k}{\Gamma_{(q)}(r+rk)} \frac{\Gamma_{(q)}(r+rk)}{\tau^{r+rk}}$$

$$= \sum_{k=0}^{\infty} \frac{\alpha^k}{\tau^{r+rk}}$$

$$= \frac{1}{\tau^r} \sum_{k=0}^{\infty} \left(\frac{\alpha}{\tau^r}\right)^k$$

$$= \frac{1}{\tau^r} \cdot \frac{\tau^r}{\tau^r - \alpha}$$

$$= \frac{1}{\tau^{r} - \alpha}.$$

The proof is completed. □

Proposition 3. *Let* $\alpha \in \mathbb{R}$, $\mathrm{Re}(r_1) > 0$ *and* $\mathrm{Re}(r_2) > 0$ *so that* $r_1 > r_2$. *Then,*

$$\mathbb{L}_q\Big(t^{r_2-1}\mathbf{E}^{q}_{r_1,r_2}(\alpha,t)\Big)(\tau) = \frac{\tau^{r_1-r_2}}{\tau^{r_1}-\alpha}, \quad \Big|\frac{\alpha}{\tau^{r_1}}\Big| < 1.$$

Proof. We know that

$$\mathbf{E}^{q}_{r_1,r_2}(\alpha,t) = \sum_{k=0}^{\infty} \frac{\alpha^k (t)^{(r_1 k)}_{(q)}}{\Gamma_{(q)}(r_2 + r_1 k)}, \quad t \in \mathbb{C}.$$

In this case, we have

$$t^{r_2-1}\mathbf{E}^{q}_{r_1,r_2}(\alpha,t) = t^{r_2-1}\sum_{k=0}^{\infty} \frac{\alpha^k (t)^{(r_1 k)}_{(q)}}{\Gamma_{(q)}(r_2 + r_1 k)}, \quad t \in \mathbb{C}. \tag{9}$$

Applying the quantum Laplace transform on (9) and by Lemma 1, we have

$$\begin{aligned}
\mathbb{L}_q\Big(t^{r_2-1}\mathbf{E}^{q}_{r_1,r_2}(\alpha,t)\Big)(\tau) &= \mathbb{L}_q\Big(t^{r_2-1}\sum_{k=0}^{\infty} \frac{\alpha^k (t)^{(r_1 k)}_{(q)}}{\Gamma_{(q)}(r_2 + r_1 k)}\Big)(\tau) \\
&= \mathbb{L}_q\Big(\sum_{k=0}^{\infty} \frac{\alpha^k t^{r_2+r_1 k-1}}{\Gamma_{(q)}(r_2 + r_1 k)}\Big)(\tau) \\
&= \sum_{k=0}^{\infty} \frac{\alpha^k}{\Gamma_{(q)}(r_2 + r_1 k)} \mathbb{L}_q\Big(t^{r_2+r_1 k-1}\Big)(\tau) \\
&= \sum_{k=0}^{\infty} \frac{\alpha^k}{\Gamma_{(q)}(r_2 + r_1 k)} \frac{\Gamma_{(q)}(r_2 + r_1 k)}{\tau^{r_2+r_1 k}} \\
&= \sum_{k=0}^{\infty} \frac{\alpha^k}{\tau^{r_2+r_1 k}} \\
&= \frac{1}{\tau^{r_2}} \sum_{k=0}^{\infty} \Big(\frac{\alpha}{\tau^{r_1}}\Big)^k \\
&= \frac{1}{\tau^{r_2}} \cdot \frac{\tau^{r_1}}{\tau^{r_1}-\alpha} \\
&= \frac{\tau^{r_1-r_2}}{\tau^{r_1}-\alpha}.
\end{aligned}$$

The proof is finally completed. □

3. Ulam–Hyers Stability of Linear q-Difference Equations

In this section, the Ulam–Hyers stability is our main focus in relation to the given linear q-difference equations of the Caputo-like type. We begin with the following theorem according to the q-difference Equation (1).

Theorem 2. *Let* $\mathtt{r} \in (0,1)$ *and* $\mathtt{q} \in (0,1)$. *Let* $\mathtt{h}$ *be a real-valued function defined on* $[0,\infty)$ *continuously. Moreover, assume that the function* $\mathtt{y} : [0,\infty) \to \mathbb{R}$ *satisfies*

$$\left|({}^{c}\mathbb{D}^{\mathtt{r}}_{(\mathtt{q})}\mathtt{y})(\mathtt{t}) - \mathtt{h}(\mathtt{t})\right| \leq \epsilon, \quad \mathtt{y}(0) = \mathtt{y}_0 \in \mathbb{R},$$

for all $\mathtt{t} \geq 0$ *and* $\epsilon > 0$. *Then, there is a solution function* $\tilde{\mathtt{y}} : [0,\infty) \to \mathbb{R}$ *of the linear* $\mathtt{q}$*-difference equation of the Caputo-like type*

$$({}^{c}\mathbb{D}^{\mathtt{r}}_{(\mathtt{q})}\mathtt{y})(\mathtt{t}) = \mathtt{h}(\mathtt{t}), \quad \mathtt{t} \geq 0,$$

so that

$$\left|\mathtt{y}(\mathtt{t}) - \tilde{\mathtt{y}}(\mathtt{t})\right| \leq \frac{\mathtt{t}^{\mathtt{r}}}{\Gamma_{(\mathtt{q})}(\mathtt{r}+1)}\epsilon. \tag{10}$$

Proof. We put

$$\mathtt{g}(\mathtt{t}) = ({}^{c}\mathbb{D}^{\mathtt{r}}_{(\mathtt{q})}\mathtt{y})(\mathtt{t}) - \mathtt{h}(\mathtt{t}), \quad \mathtt{t} \geq 0. \tag{11}$$

By Lemma 3, we use the quantum Laplace transform on both sides of (11) and obtain

$$\begin{aligned}
\mathbb{L}_{\mathtt{q}}\Big(\mathtt{g}(\mathtt{t})\Big)(\tau) &= \mathbb{L}_{\mathtt{q}}\Big(({}^{c}\mathbb{D}^{\mathtt{r}}_{(\mathtt{q})}\mathtt{y})(\mathtt{t}) - \mathtt{h}(\mathtt{t})\Big)(\tau) \\
&= \mathbb{L}_{\mathtt{q}}\Big(({}^{c}\mathbb{D}^{\mathtt{r}}_{(\mathtt{q})}\mathtt{y})(\mathtt{t})\Big)(\tau) - \mathbb{L}_{\mathtt{q}}\Big(\mathtt{h}(\mathtt{t})\Big)(\tau) \\
&= \tau^{\mathtt{r}}\mathbb{L}_{\mathtt{q}}\Big(\mathtt{y}(\mathtt{t})\Big)(\tau) - \tau^{\mathtt{r}-1}\mathtt{y}_0 - \mathbb{L}_{\mathtt{q}}\Big(\mathtt{h}(\mathtt{t})\Big)(\tau).
\end{aligned} \tag{12}$$

Note that we denote the quantum Laplace transform of g by $\mathbb{L}_{\mathtt{q}}\Big(\mathtt{g}(\cdot)\Big)$.
In the following, (12) implies that

$$\mathbb{L}_{\mathtt{q}}\Big(\mathtt{y}(\mathtt{t})\Big)(\tau) = \frac{1}{\tau^{\mathtt{r}}}\mathbb{L}_{\mathtt{q}}\Big(\mathtt{g}(\mathtt{t})\Big)(\tau) + \frac{1}{\tau^{\mathtt{r}}}\mathbb{L}_{\mathtt{q}}\Big(\mathtt{h}(\mathtt{t})\Big)(\tau) + \frac{1}{\tau}\mathtt{y}_0.$$

Hence, we can write

$$\begin{aligned}
\mathbb{L}_{\mathtt{q}}\Big(\mathtt{y}(\mathtt{t})\Big)(\tau) &= \frac{1}{\Gamma_{(\mathtt{q})}(\mathtt{r})}\mathbb{L}_{\mathtt{q}}\Big(\mathtt{t}^{\mathtt{r}-1}\Big)(\tau)\mathbb{L}_{\mathtt{q}}\Big(\mathtt{g}(\mathtt{t})\Big)(\tau) \\
&\quad + \frac{1}{\Gamma_{(\mathtt{q})}(\mathtt{r})}\mathbb{L}_{\mathtt{q}}\Big(\mathtt{t}^{\mathtt{r}-1}\Big)(\tau)\mathbb{L}_{\mathtt{q}}\Big(\mathtt{h}(\mathtt{t})\Big)(\tau) + \mathtt{y}_0\mathbb{L}_{\mathtt{q}}\Big(1\Big)(\tau) \\
&= \frac{1}{\Gamma_{(\mathtt{q})}(\mathtt{r})}\mathbb{L}_{\mathtt{q}}\Big(\mathtt{t}^{\mathtt{r}-1} *_{\mathtt{q}} \mathtt{g}(\mathtt{t})\Big)(\tau) \\
&\quad + \frac{1}{\Gamma_{(\mathtt{q})}(\mathtt{r})}\mathbb{L}_{\mathtt{q}}\Big(\mathtt{t}^{\mathtt{r}-1} *_{\mathtt{q}} \mathtt{h}(\mathtt{t})\Big)(\tau) + \mathtt{y}_0\mathbb{L}_{\mathtt{q}}\Big(1\Big)(\tau).
\end{aligned}$$

We define

$$\begin{aligned}
\tilde{\mathtt{y}}(\mathtt{t}) &= \mathtt{y}_0 + \frac{1}{\Gamma_{(\mathtt{q})}(\mathtt{r})}\Big(\mathtt{t}^{\mathtt{r}-1} *_{\mathtt{q}} \mathtt{h}(\mathtt{t})\Big) \\
&= \mathtt{y}_0 + \frac{1}{\Gamma_{(\mathtt{q})}(\mathtt{r})}\int_0^{\mathtt{t}} (\mathtt{t} - \mathtt{q}\mathtt{w})^{(\mathtt{r}-1)}_{(\mathtt{q})}\mathtt{h}(\mathtt{w})\, d_{(\mathtt{q})}\mathtt{w}.
\end{aligned} \tag{13}$$

On both sides of (13), we apply the quantum Laplace transform. Thus, we obtain

$$\mathbb{L}_{\mathtt{q}}\Big(\tilde{\mathtt{y}}(\mathtt{t})\Big)(\tau) = \frac{1}{\tau}\mathtt{y}_0 + \frac{1}{\Gamma_{(\mathtt{q})}(\mathtt{r})}\mathbb{L}_{\mathtt{q}}\Big(\int_0^{\mathtt{t}} (\mathtt{t} - \mathtt{q}\mathtt{w})^{(\mathtt{r}-1)}_{(\mathtt{q})}\mathtt{h}(\mathtt{w})\, d_{(\mathtt{q})}\mathtt{w}\Big)(\tau)$$

$$= \frac{1}{\tau} y_0 + \frac{1}{\Gamma_{(q)}(r)} \mathbb{L}_q \Big(t^{r-1} *_q h(t) \Big)(\tau)$$

$$= \frac{1}{\tau} y_0 + \mathbb{L}_q \Big(\frac{1}{\Gamma_{(q)}(r)} t^{r-1} \Big)(\tau) \cdot \mathbb{L}_q \Big(h(t) \Big)(\tau)$$

$$= \frac{1}{\tau} y_0 + \frac{1}{\tau^r} \mathbb{L}_q \Big(h(t) \Big)(\tau).$$

Therefore,

$$\mathbb{L}_q \Big(\tilde{y}(t) \Big)(\tau) = \frac{1}{\tau} y_0 + \frac{1}{\tau^r} \mathbb{L}_q \Big(h(t) \Big)(\tau). \tag{14}$$

On the other side, we compute the following equalities:

$$\mathbb{L}_q \Big(({}^c\mathbb{D}^r_{(q)} \tilde{y})(t) \Big)(\tau) = \tau^r \mathbb{L}_q \Big(\tilde{y}(t) \Big)(\tau) - \tau^{r-1} y_0$$

$$= \tau^{r-1} y_0 + \mathbb{L}_q \Big(h(t) \Big)(\tau) - \tau^{r-1} y_0$$

$$= \mathbb{L}_q \Big(h(t) \Big)(\tau).$$

The latter equalities show that the function $\tilde{y}(\cdot)$ satisfies the linear q-difference Equation (1) of the Caputo-like type. In this step, (13) and (14) give

$$\mathbb{L}_q \Big(y(t) - \tilde{y}(t) \Big)(\tau) = \frac{1}{\tau^r} \mathbb{L}_q \Big(g(t) \Big)(\tau) + \frac{1}{\tau^r} \mathbb{L}_q \Big(h(t) \Big)(\tau)$$

$$+ \frac{1}{\tau} y_0 - \frac{1}{\tau} y_0 - \frac{1}{\tau^r} \mathbb{L}_q \Big(h(t) \Big)(\tau)$$

$$= \frac{1}{\tau^r} \mathbb{L}_q \Big(g(t) \Big)(\tau)$$

$$= \mathbb{L}_q \Big(\frac{1}{\Gamma_{(q)}(r)} t^{r-1} \Big)(\tau) \cdot \mathbb{L}_q \Big(g(t) \Big)(\tau)$$

$$= \mathbb{L}_q \Big(\frac{1}{\Gamma_{(q)}(r)} t^{r-1} *_q g(t) \Big)(\tau)$$

$$= \mathbb{L}_q \Big(\int_0^t \frac{1}{\Gamma_{(q)}(r)} (t - qw)^{(r-1)}_{(q)} g(w) \, d_{(q)} w \Big)(\tau).$$

Since $\mathbb{L}_q$ is a one-to-one transformation, we have

$$y(t) - \tilde{y}(t) = \frac{1}{\Gamma_{(q)}(r)} \int_0^t (t - qw)^{(r-1)}_{(q)} g(w) \, d_{(q)} w.$$

Therefore, we estimate

$$\big| y(t) - \tilde{y}(t) \big| \le \frac{1}{\Gamma_{(q)}(r)} \int_0^t (t - qw)^{(r-1)}_{(q)} \big| g(w) \big| \, d_{(q)} w$$

$$\le \frac{\epsilon}{\Gamma_{(q)}(r)} \int_0^t (t - qw)^{(r-1)}_{(q)} d_{(q)} w$$

$$= \frac{t^r}{\Gamma_{(q)}(r+1)} \epsilon,$$

and this completes the proof. □

In view of Definition 1, the inequality (10) guarantees that the linear q-difference equation of the Caputo-like type (1) is Ulam–Hyers stable with the corresponding stability constant

$$K = \frac{\mathtt{T}^{\mathtt{r}}}{\Gamma_{(\mathtt{q})}(\mathtt{r}+1)},$$

so that $0 \leq \mathtt{t} \leq \mathtt{T}$. It is clear that (1) is not Ulam–Hyers stable if $\mathtt{t} = \infty$.

Now, we continue this study on the stability of the second linear q-difference Equation (2) of the Caputo-like type by stating the next theorem.

Theorem 3. *Let* $\mathtt{r} \in (0,1)$, $\mathtt{q} \in (0,1)$, $\alpha \in \mathbb{R}$, *and* $\mathtt{h}$ *be a real-valued function defined on* $[0,\infty)$ *continuously. Also, assume that the function* $\mathtt{y} : [0,\infty) \to \mathbb{R}$ *satisfies*

$$\left|({}^{c}\mathbb{D}^{\mathtt{r}}_{(\mathtt{q})}\mathtt{y})(\mathtt{t}) - \alpha\mathtt{y}(\mathtt{t}) - \mathtt{h}(\mathtt{t})\right| \leq \epsilon, \quad \mathtt{y}(0) = \mathtt{y}_0 \in \mathbb{R},$$

for all $\mathtt{t} \geq 0$ *and* $\epsilon > 0$. *Then, a solution function* $\tilde{\mathtt{y}} : [0,\infty) \to \mathbb{R}$ *exists for the linear* q*-difference equation of the Caputo-like type*

$$({}^{c}\mathbb{D}^{\mathtt{r}}_{(\mathtt{q})}\mathtt{y})(\mathtt{t}) - \alpha\mathtt{y}(\mathtt{t}) = \mathtt{h}(\mathtt{t}), \quad \mathtt{t} \geq 0,$$

so that

$$\left|\mathtt{y}(\mathtt{t}) - \tilde{\mathtt{y}}(\mathtt{t})\right| \leq \mathtt{t}^{\mathtt{r}}\mathbf{E}^{\mathtt{q}}_{\mathtt{r},\mathtt{r}+1}(\alpha,\mathtt{t})\epsilon. \tag{15}$$

Proof. We first consider

$$\mathtt{g}_1(\mathtt{t}) = ({}^{c}\mathbb{D}^{\mathtt{r}}_{(\mathtt{q})}\mathtt{y})(\mathtt{t}) - \alpha\mathtt{y}(\mathtt{t}) - \mathtt{h}(\mathtt{t}), \tag{16}$$

for all $\mathtt{t} \geq 0$. By Lemma 3, we use the quantum Laplace transform on both sides of (16) and obtain

$$\begin{aligned}
\mathbb{L}_{\mathtt{q}}\Big(\mathtt{g}_1(\mathtt{t})\Big)(\tau) &= \mathbb{L}_{\mathtt{q}}\Big(({}^{c}\mathbb{D}^{\mathtt{r}}_{(\mathtt{q})}\mathtt{y})(\mathtt{t}) - \alpha\mathtt{y}(\mathtt{t}) - \mathtt{h}(\mathtt{t})\Big)(\tau) \\
&= \mathbb{L}_{\mathtt{q}}\Big(({}^{c}\mathbb{D}^{\mathtt{r}}_{(\mathtt{q})}\mathtt{y})(\mathtt{t})\Big)(\tau) - \alpha\mathbb{L}_{\mathtt{q}}\Big(\mathtt{y}(\mathtt{t})\Big)(\tau) - \mathbb{L}_{\mathtt{q}}\Big(\mathtt{h}(\mathtt{t})\Big)(\tau) \\
&= \tau^{\mathtt{r}}\mathbb{L}_{\mathtt{q}}\Big(\mathtt{y}(\mathtt{t})\Big)(\tau) - \tau^{\mathtt{r}-1}\mathtt{y}_0 - \alpha\mathbb{L}_{\mathtt{q}}\Big(\mathtt{y}(\mathtt{t})\Big)(\tau) - \mathbb{L}_{\mathtt{q}}\Big(\mathtt{h}(\mathtt{t})\Big)(\tau).
\end{aligned} \tag{17}$$

Here, the quantum Laplace transform of $\mathtt{g}_1$ is denoted by $\mathbb{L}_{\mathtt{q}}\Big(\mathtt{g}_1(\cdot)\Big)$.

In the next step, by (17), we can write

$$\begin{aligned}
\mathbb{L}_{\mathtt{q}}\Big(\mathtt{y}(\mathtt{t})\Big)(\tau) &= \frac{1}{\tau^{\mathtt{r}} - \alpha}\mathbb{L}_{\mathtt{q}}\Big(\mathtt{g}_1(\mathtt{t})\Big)(\tau) + \frac{1}{\tau^{\mathtt{r}} - \alpha}\mathbb{L}_{\mathtt{q}}\Big(\mathtt{h}(\mathtt{t})\Big)(\tau) + \frac{\tau^{\mathtt{r}-1}}{\tau^{\mathtt{r}} - \alpha}\mathtt{y}_0 \\
&= \mathbb{L}_{\mathtt{q}}\Big(\mathtt{t}^{\mathtt{r}-1}\mathbf{E}^{\mathtt{q}}_{\mathtt{r},\mathtt{r}}(\alpha,\mathtt{t})\Big)(\tau) \cdot \mathbb{L}_{\mathtt{q}}\Big(\mathtt{g}_1(\mathtt{t})\Big)(\tau) \\
&+ \mathbb{L}_{\mathtt{q}}\Big(\mathtt{t}^{\mathtt{r}-1}\mathbf{E}^{\mathtt{q}}_{\mathtt{r},\mathtt{r}}(\alpha,\mathtt{t})\Big)(\tau) \cdot \mathbb{L}_{\mathtt{q}}\Big(\mathtt{h}(\mathtt{t})\Big)(\tau) + \mathtt{y}_0\mathbb{L}_{\mathtt{q}}\Big(\mathbf{E}^{\mathtt{q}}_{\mathtt{r}}(\alpha,\mathtt{t})\Big)(\tau) \\
&= \mathbb{L}_{\mathtt{q}}\Big(\mathtt{t}^{\mathtt{r}-1}\mathbf{E}^{\mathtt{q}}_{\mathtt{r},\mathtt{r}}(\alpha,\mathtt{t}) *_{\mathtt{q}} \mathtt{g}_1(\mathtt{t})\Big)(\tau) \\
&+ \mathbb{L}_{\mathtt{q}}\Big(\mathtt{t}^{\mathtt{r}-1}\mathbf{E}^{\mathtt{q}}_{\mathtt{r},\mathtt{r}}(\alpha,\mathtt{t}) *_{\mathtt{q}} \mathtt{h}(\mathtt{t})\Big)(\tau) + \mathtt{y}_0\mathbb{L}_{\mathtt{q}}\Big(\mathbf{E}^{\mathtt{q}}_{\mathtt{r}}(\alpha,\mathtt{t})\Big)(\tau) \\
&= \mathbb{L}_{\mathtt{q}}\Big(\int_0^{\mathtt{t}}(\mathtt{t} - \mathtt{q}\mathtt{w})^{(\mathtt{r}-1)}_{(\mathtt{q})}\mathbf{E}^{\mathtt{q}}_{\mathtt{r},\mathtt{r}}(\alpha,\mathtt{t} - \mathtt{q}\mathtt{w}) \cdot \mathtt{g}_1(\mathtt{w})\, d_{(\mathtt{q})}\mathtt{w}\Big)(\tau)
\end{aligned}$$

$$+\mathbb{L}_q\Big(\int_0^t (t-qw)_{(q)}^{(r-1)}\mathbf{E}_{r,r}^q(\alpha, t-qw)\cdot h(w)\, d_{(q)}w\Big)(\tau)$$
$$+ y_0\mathbb{L}_q\Big(\mathbf{E}_r^q(\alpha, t)\Big)(\tau). \tag{18}$$

Define

$$\tilde{y}(t) = y_0\mathbf{E}_r^q(\alpha, t) + \int_0^t (t-qw)_{(q)}^{(r-1)}\mathbf{E}_{r,r}^q(\alpha, t-qw)\cdot h(w)\, d_{(q)}w. \tag{19}$$

Now, we apply the quantum Laplace transform on (19) as follows:

$$\begin{aligned}
\mathbb{L}_q\Big(\tilde{y}(t)\Big)(\tau) &= y_0\mathbb{L}_q\Big(\mathbf{E}_r^q(\alpha, t)\Big)(\tau) \\
&+\mathbb{L}_q\Big(\int_0^t (t-qw)_{(q)}^{(r-1)}\mathbf{E}_{r,r}^q(\alpha, t-qw)\cdot h(w)\, d_{(q)}w\Big)(\tau) \\
&= y_0\mathbb{L}_q\Big(\mathbf{E}_r^q(\alpha, t)\Big)(\tau) \\
&+\mathbb{L}_q\Big(t^{r-1}\mathbf{E}_{r,r}^q(\alpha, t) *_q h(t)\Big)(\tau) \\
&= y_0\mathbb{L}_q\Big(\mathbf{E}_r^q(\alpha, t)\Big)(\tau) \\
&+\mathbb{L}_q\Big(t^{r-1}\mathbf{E}_{r,r}^q(\alpha, t)\Big)(\tau)\cdot\mathbb{L}_q\Big(h(t)\Big)(\tau) \\
&= \frac{\tau^{r-1}}{\tau^r-\alpha}y_0 + \frac{1}{\tau^r-\alpha}\mathbb{L}_q\Big(h(t)\Big)(\tau).
\end{aligned} \tag{20}$$

Lemma 3 and (20) imply that

$$\begin{aligned}
\mathbb{L}_q\Big(({}^c\mathbb{D}_{(q)}^r\tilde{y})(t) - \alpha\tilde{y}(t)\Big)(\tau) &= \tau^r\mathbb{L}_q\Big(\tilde{y}(t)\Big)(\tau) - \tau^{r-1}y_0 - \alpha\mathbb{L}_q\Big(\tilde{y}(t)\Big)(\tau) \\
&= (\tau^r-\alpha)\frac{\tau^{r-1}}{\tau^r-\alpha}y_0 + (\tau^r-\alpha)\frac{1}{\tau^r-\alpha}\mathbb{L}_q\Big(h(t)\Big)(\tau) - \tau^{r-1}y_0 \\
&= \mathbb{L}_q\Big(h(t)\Big)(\tau),
\end{aligned}$$

which shows that the newly defined function $\tilde{y}$ is a solution of (2).

In this step, in view of (18) and (20), we have

$$\begin{aligned}
\mathbb{L}_q\Big(y(t)-\tilde{y}(t)\Big)(\tau) &= \mathbb{L}_q\Big(\int_0^t (t-qw)_{(q)}^{(r-1)}\mathbf{E}_{r,r}^q(\alpha, t-qw)\cdot g_1(w)\, d_{(q)}w\Big)(\tau) \\
&+\mathbb{L}_q\Big(\int_0^t (t-qw)_{(q)}^{(r-1)}\mathbf{E}_{r,r}^q(\alpha, t-qw)\cdot h(w)\, d_{(q)}w\Big)(\tau) \\
&+ y_0\mathbb{L}_q\Big(\mathbf{E}_r^q(\alpha, t)\Big)(\tau) - y_0\mathbb{L}_q\Big(\mathbf{E}_r^q(\alpha, t)\Big)(\tau) \\
&-\mathbb{L}_q\Big(\int_0^t (t-qw)_{(q)}^{(r-1)}\mathbf{E}_{r,r}^q(\alpha, t-qw)\cdot h(w)\, d_{(q)}w\Big)(\tau) \\
&= \mathbb{L}_q\Big(\int_0^t (t-qw)_{(q)}^{(r-1)}\mathbf{E}_{r,r}^q(\alpha, t-qw)\cdot g_1(w)\, d_{(q)}w\Big)(\tau).
\end{aligned}$$

Therefore,

$$y(t)-\tilde{y}(t) = \int_0^t (t-qw)_{(q)}^{(r-1)}\mathbf{E}_{r,r}^q(\alpha, t-qw)\cdot g_1(w)\, d_{(q)}w.$$

Accordingly, we estimate

$$\begin{aligned}
\left|y(t)-\tilde{y}(t)\right| &= \left|\int_0^t (t-qw)_{(q)}^{(r-1)} \mathbf{E}_{r,r}^{q}(\alpha, t-qw)\cdot g_1(w)\, d_{(q)}w\right| \\
&\leq \int_0^t (t-qw)_{(q)}^{(r-1)} \mathbf{E}_{r,r}^{q}(\alpha, t-qw)\left|g_1(w)\right| d_{(q)}w \\
&\leq \epsilon \int_0^t (t-qw)_{(q)}^{(r-1)} \mathbf{E}_{r,r}^{q}(\alpha, t-qw)\, d_{(q)}w \\
&= \epsilon \int_0^t (t-qw)_{(q)}^{(r-1)} \sum_{k=0}^{\infty} \frac{\alpha^k (t-qw)_{(q)}^{(kr)}}{\Gamma_{(q)}(kr+r)}\, d_{(q)}w \\
&= \epsilon \sum_{k=0}^{\infty} \frac{\alpha^k}{\Gamma_{(q)}(kr+r)} \int_0^t (t-qw)_{(q)}^{(kr+r-1)}\, d_{(q)}w \\
&= \epsilon \sum_{k=0}^{\infty} \frac{\alpha^k}{\Gamma_{(q)}(kr+r)} \cdot \frac{t^{kr+r}}{(kr+r)} \\
&= \epsilon t^r \sum_{k=0}^{\infty} \frac{\alpha^k (t)_{(q)}^{(kr)}}{\Gamma_{(q)}(kr+r+1)} \\
&= \epsilon t^r \mathbf{E}_{r,r+1}^{q}(\alpha, t).
\end{aligned}$$

This completes the proof. □

In view of Definition 1, the inequality (15) implies that the linear q-difference Equation (2) of the Caputo-like type is Ulam–Hyers stable with the corresponding stability constant

$$K = T^r \mathbf{E}_{r,r+1}^{q}(\alpha, T),$$

so that $0 \leq t \leq T$. Clearly, (2) is not Ulam–Hyers stable if $t = \infty$.

In the next theorem, the stability property is analyzed for the third linear q-difference Equation (3) of the Caputo-like type.

Theorem 4. *Let $r_1, r_2 \in (0,1)$ with $r_2 < r_1$, $q \in (0,1)$, and $\alpha \in \mathbb{R}$. Let the real-valued function h be continuous on $[0,\infty)$ and the function $y : [0,\infty) \to \mathbb{R}$ satisfies*

$$\left|({}^c\mathbb{D}_{(q)}^{r_1} y)(t) - \alpha({}^c\mathbb{D}_{(q)}^{r_2} y)(t) - h(t)\right| \leq \epsilon, \quad y(0) = y_0 \in \mathbb{R},$$

for all $t \geq 0$ and $\epsilon > 0$. Then, a solution function $\tilde{y} : [0,\infty) \to \mathbb{R}$ exists for the linear q-difference equation of the Caputo-like type

$$({}^c\mathbb{D}_{(q)}^{r_1} y)(t) - \alpha({}^c\mathbb{D}_{(q)}^{r_2} y)(t) = h(t), \quad t \geq 0,$$

so that

$$\left|y(t) - \tilde{y}(t)\right| \leq t^{r_1} \mathbf{E}_{r_1-r_2, r_1+1}^{q}(\alpha, t)\epsilon. \tag{21}$$

Proof. We set

$$g_2(t) = ({}^c\mathbb{D}_{(q)}^{r_1} y)(t) - \alpha({}^c\mathbb{D}_{(q)}^{r_2} y)(t) - h(t), \tag{22}$$

for all $t \geq 0$. On both sides of (22), we apply the quantum Laplace transform. So, by Lemma 3, we have

$$
\begin{aligned}
\mathbb{L}_q\Big(g_2(t)\Big)(\tau) &= \mathbb{L}_q\Big(({}^{c}\mathbb{D}^{r_1}_{(q)}y)(t) - \alpha({}^{c}\mathbb{D}^{r_2}_{(q)}y)(t) - h(t)\Big)(\tau)\\
&= \mathbb{L}_q\Big(({}^{c}\mathbb{D}^{r_1}_{(q)}y)(t)\Big)(\tau) - \alpha\mathbb{L}_q\Big(({}^{c}\mathbb{D}^{r_2}_{(q)}y)(t)\Big)(\tau) - \mathbb{L}_q\Big(h(t)\Big)(\tau)\\
&= \tau^{r_1}\mathbb{L}_q\Big(y(t)\Big)(\tau) - \tau^{r_1-1}y_0 - \alpha\tau^{r_2}\mathbb{L}_q\Big(y(t)\Big)(\tau)\\
&+ \alpha\tau^{r_2-1}y_0 - \mathbb{L}_q\Big(h(t)\Big)(\tau)\\
&= (\tau^{r_1} - \alpha\tau^{r_2})\mathbb{L}_q\Big(y(t)\Big)(\tau) - (\tau^{r_1-1} - \alpha\tau^{r_2-1})y_0 - \mathbb{L}_q\Big(h(t)\Big)(\tau), \quad (23)
\end{aligned}
$$

where the symbol $\mathbb{L}_q\Big(g_2(\cdot)\Big)$ is the quantum Laplace transform of the function g_2.

Notice that

$$
\frac{\tau^{r_1-1}}{\tau^{r_1} - \alpha\tau^{r_2}} = \frac{\tau^{r_1-r_2-1}}{\tau^{r_1-r_2} - \alpha} = \mathbb{L}_q\Big(\mathbf{E}^q_{r_1-r_2}(\alpha, t)\Big)(\tau);
$$

$$
\frac{\tau^{r_2-1}}{\tau^{r_1} - \alpha\tau^{r_2}} = \frac{\tau^{-1}}{\tau^{r_1-r_2} - \alpha} = \frac{\tau^{(r_1-r_2)-(r_1-r_2+1)}}{\tau^{r_1-r_2} - \alpha} = \mathbb{L}_q\Big(t^{r_1-r_2}\mathbf{E}^q_{r_1-r_2, r_1-r_2+1}(\alpha, t)\Big)(\tau);
$$

and

$$
\frac{1}{(\tau^{r_1} - \alpha\tau^{r_2})} = \frac{\tau^{-r_2}}{\tau^{r_1-r_2} - \alpha} = \frac{\tau^{(r_1-r_2)-r_1}}{\tau^{r_1-r_2} - \alpha} = \mathbb{L}_q\Big(t^{r_1-1}\mathbf{E}^q_{r_1-r_2, r_1}(\alpha, t)\Big)(\tau).
$$

By considering (23), one can write

$$
\begin{aligned}
\mathbb{L}_q\Big(y(t)\Big)(\tau) &= \frac{\tau^{r_1-1} - \alpha\tau^{r_2-1}}{\tau^{r_1} - \alpha\tau^{r_2}}y_0\\
&+ \frac{1}{\tau^{r_1} - \alpha\tau^{r_2}}\mathbb{L}_q\Big(h(t)\Big)(\tau) + \frac{1}{\tau^{r_1} - \alpha\tau^{r_2}}\mathbb{L}_q\Big(g_2(t)\Big)(\tau)\\
&= \frac{\tau^{r_1-1}}{\tau^{r_1} - \alpha\tau^{r_2}}y_0 - \alpha\frac{\tau^{r_2-1}}{\tau^{r_1} - \alpha\tau^{r_2}}y_0\\
&+ \frac{1}{\tau^{r_1} - \alpha\tau^{r_2}}\mathbb{L}_q\Big(h(t)\Big)(\tau) + \frac{1}{\tau^{r_1} - \alpha\tau^{r_2}}\mathbb{L}_q\Big(g_2(t)\Big)(\tau)\\
&= y_0\mathbb{L}_q\Big(\mathbf{E}^q_{r_1-r_2}(\alpha, t)\Big)(\tau) - \alpha y_0\mathbb{L}_q\Big(t^{r_1-r_2}\mathbf{E}^q_{r_1-r_2, r_1-r_2+1}(\alpha, t)\Big)(\tau)\\
&+ \mathbb{L}_q\Big(t^{r_1-1}\mathbf{E}^q_{r_1-r_2, r_1}(\alpha, t)\Big)(\tau) \cdot \mathbb{L}_q\Big(h(t)\Big)(\tau)\\
&+ \mathbb{L}_q\Big(t^{r_1-1}\mathbf{E}^q_{r_1-r_2, r_1}(\alpha, t)\Big)(\tau) \cdot \mathbb{L}_q\Big(g_2(t)\Big)(\tau)\\
&= y_0\mathbb{L}_q\Big(\mathbf{E}^q_{r_1-r_2}(\alpha, t)\Big)(\tau) - \alpha y_0\mathbb{L}_q\Big(t^{r_1-r_2}\mathbf{E}^q_{r_1-r_2, r_1-r_2+1}(\alpha, t)\Big)(\tau)\\
&+ \mathbb{L}_q\Big(t^{r_1-1}\mathbf{E}^q_{r_1-r_2, r_1}(\alpha, t) *_q h(t)\Big)(\tau)\\
&+ \mathbb{L}_q\Big(t^{r_1-1}\mathbf{E}^q_{r_1-r_2, r_1}(\alpha, t) *_q g_2(t)\Big)(\tau)\\
&= y_0\mathbb{L}_q\Big(\mathbf{E}^q_{r_1-r_2}(\alpha, t)\Big)(\tau) - \alpha y_0\mathbb{L}_q\Big(t^{r_1-r_2}\mathbf{E}^q_{r_1-r_2, r_1-r_2+1}(\alpha, t)\Big)(\tau)\\
&+ \mathbb{L}_q\Big(\int_0^t (t - qw)^{(r_1-1)}_{(q)}\mathbf{E}^q_{r_1-r_2, r_1}(\alpha, t - qw) \cdot h(w)\, d_{(q)}w\Big)(\tau)
\end{aligned}
$$

$$+\mathbb{L}_q\Big(\int_0^t (t-qw)_{(q)}^{(r_1-1)}\mathbf{E}^q_{r_1-r_2,r_1}(\alpha,t-qw)\cdot g_2(w)\,d_{(q)}w\Big)(\tau). \tag{24}$$

Define

$$\tilde{y}(t) = y_0\mathbf{E}^q_{r_1-r_2}(\alpha,t) - \alpha y_0 t^{r_1-r_2}\mathbf{E}^q_{r_1-r_2,r_1-r_2+1}(\alpha,t) + \int_0^t (t-qw)_{(q)}^{(r_1-1)}\mathbf{E}^q_{r_1-r_2,r_1}(\alpha,t-qw)\cdot h(w)\,d_{(q)}w. \tag{25}$$

Now, we take the quantum Laplace transform of $\tilde{y}(t)$ as follows:

$$\begin{aligned}
\mathbb{L}_q\Big(\tilde{y}(t)\Big)(\tau) &= y_0\mathbb{L}_q\Big(\mathbf{E}^q_{r_1-r_2}(\alpha,t)\Big)(\tau) - \alpha y_0\mathbb{L}_q\Big(t^{r_1-r_2}\mathbf{E}^q_{r_1-r_2,r_1-r_2+1}(\alpha,t)\Big)(\tau)\\
&+\mathbb{L}_q\Big(\int_0^t (t-qw)_{(q)}^{(r_1-1)}\mathbf{E}^q_{r_1-r_2,r_1}(\alpha,t-qw)\cdot h(w)\,d_{(q)}w\Big)(\tau)\\
&= y_0\mathbb{L}_q\Big(\mathbf{E}^q_{r_1-r_2}(\alpha,t)\Big)(\tau) - \alpha y_0\mathbb{L}_q\Big(t^{r_1-r_2}\mathbf{E}^q_{r_1-r_2,r_1-r_2+1}(\alpha,t)\Big)(\tau)\\
&+\mathbb{L}_q\Big(t^{r_1-1}\mathbf{E}^q_{r_1-r_2,r_1}(\alpha,t) *_q h(t)\Big)(\tau)\\
&= \frac{\tau^{r_1-1}}{\tau^{r_1}-\alpha\tau^{r_2}}y_0 - \alpha\frac{\tau^{r_2-1}}{\tau^{r_1}-\alpha\tau^{r_2}}y_0 + \frac{1}{\tau^{r_1}-\alpha\tau^{r_2}}\mathbb{L}_q\Big(h(t)\Big)(\tau)\\
&= \frac{\tau^{r_1-1}-\alpha\tau^{r_2-1}}{\tau^{r_1}-\alpha\tau^{r_2}}y_0 + \frac{1}{\tau^{r_1}-\alpha\tau^{r_2}}\mathbb{L}_q\Big(h(t)\Big)(\tau).
\end{aligned} \tag{26}$$

Now, the definition of the quantum Laplace transform for the fractional quantum derivative of the Caputo-like type and (26) imply that

$$\begin{aligned}
\mathbb{L}_q\Big(({}^c\mathbb{D}^{r_1}_{(q)}\tilde{y})(t) - \alpha({}^c\mathbb{D}^{r_2}_{(q)}\tilde{y})(t)\Big)(\tau) &= \tau^{r_1}\mathbb{L}_q\Big(\tilde{y}(t)\Big)(\tau) - \tau^{r_1-1}y_0\\
&- \alpha\tau^{r_2}\mathbb{L}_q\Big(\tilde{y}(t)\Big)(\tau) + \alpha\tau^{r_2-1}y_0\\
&= (\tau^{r_1}-\alpha\tau^{r_2})\mathbb{L}_q\Big(\tilde{y}(t)\Big)(\tau) - (\tau^{r_1-1}-\alpha\tau^{r_2-1})y_0\\
&= (\tau^{r_1}-\alpha\tau^{r_2})\frac{\tau^{r_1-1}-\alpha\tau^{r_2-1}}{\tau^{r_1}-\alpha\tau^{r_2}}y_0\\
&+ (\tau^{r_1}-\alpha\tau^{r_2})\frac{1}{\tau^{r_1}-\alpha\tau^{r_2}}\mathbb{L}_q\Big(h(t)\Big)(\tau) - (\tau^{r_1-1}-\alpha\tau^{r_2-1})y_0\\
&= \mathbb{L}_q\Big(h(t)\Big)(\tau).
\end{aligned}$$

Since $\mathbb{L}_q$ is one-to-one, the latter equalities show that the function $\tilde{y}$ satisfies (3). On the other hand, by (24) and (26), one can write

$$\begin{aligned}
\mathbb{L}_q\Big(y(t)-\tilde{y}(t)\Big)(\tau) &= \mathbb{L}_q\Big(y(t)\Big)(\tau) - \mathbb{L}_q\Big(\tilde{y}(t)\Big)(\tau)\\
&= \frac{\tau^{r_1-1}-\alpha\tau^{r_2-1}}{\tau^{r_1}-\alpha\tau^{r_2}}y_0 + \frac{1}{\tau^{r_1}-\alpha\tau^{r_2}}\mathbb{L}_q\Big(h(t)\Big)(\tau)\\
&+ \frac{1}{\tau^{r_1}-\alpha\tau^{r_2}}\mathbb{L}_q\Big(g_2(t)\Big)(\tau) - \frac{\tau^{r_1-1}-\alpha\tau^{r_2-1}}{\tau^{r_1}-\alpha\tau^{r_2}}y_0\\
&- \frac{1}{\tau^{r_1}-\alpha\tau^{r_2}}\mathbb{L}_q\Big(h(t)\Big)(\tau)
\end{aligned}$$

$$= \frac{1}{\tau^{r_1} - \alpha\tau^{r_2}} \mathbb{L}_q\Big(g_2(t)\Big)(\tau)$$
$$= \mathbb{L}_q\Big(t^{r_1-1}\mathbf{E}^q_{r_1-r_2,r_1}(\alpha,t)\Big)(\tau)\cdot\mathbb{L}_q\Big(g_2(t)\Big)(\tau)$$
$$= \mathbb{L}_q\Big(\int_0^t (t-qw)_{(q)}^{(r_1-1)}\mathbf{E}^q_{r_1-r_2,r_1}(\alpha,t-qw)\cdot g_2(w)\,d_{(q)}w\Big)(\tau).$$

This gives that

$$y(t)-\tilde{y}(t) = \int_0^t (t-qw)_{(q)}^{(r_1-1)}\mathbf{E}^q_{r_1-r_2,r_1}(\alpha,t-qw)\cdot g_2(w)\,d_{(q)}w.$$

Hence, we have

$$\begin{aligned}
\big|y(t)-\tilde{y}(t)\big| &= \left|\int_0^t (t-qw)_{(q)}^{(r_1-1)}\mathbf{E}^q_{r_1-r_2,r_1}(\alpha,t-qw)\cdot g_2(w)\,d_{(q)}w\right| \\
&\le \int_0^t (t-qw)_{(q)}^{(r_1-1)}\mathbf{E}^q_{r_1-r_2,r_1}(\alpha,t-qw)\big|g_2(w)\big|\,d_{(q)}w \\
&\le \epsilon\int_0^t (t-qw)_{(q)}^{(r_1-1)}\mathbf{E}^q_{r_1-r_2,r_1}(\alpha,t-qw)d_{(q)}w \\
&= \epsilon\int_0^t (t-qw)_{(q)}^{(r_1-1)}\sum_{k=0}^{\infty}\frac{\alpha^k(t-qw)_{(q)}^{(k(r_1-r_2))}}{\Gamma_{(q)}(k(r_1-r_2)+r_1)}d_{(q)}w \\
&= \epsilon\sum_{k=0}^{\infty}\frac{\alpha^k}{\Gamma_{(q)}(k(r_1-r_2)+r_1)}\int_0^t (t-qw)_{(q)}^{(k(r_1-r_2)+r_1-1)}d_{(q)}w \\
&= \epsilon\sum_{k=0}^{\infty}\frac{\alpha^k}{\Gamma_{(q)}(k(r_1-r_2)+r_1)}\cdot\frac{t^{k(r_1-r_2)+r_1}}{(k(r_1-r_2)+r_1)} \\
&= \epsilon\sum_{k=0}^{\infty}\frac{\alpha^k t^{k(r_1-r_2)+r_1}}{\Gamma_{(q)}(k(r_1-r_2)+r_1+1)} \\
&= \epsilon t^{r_1}\sum_{k=0}^{\infty}\frac{\alpha^k(t)_{(q)}^{(k(r_1-r_2))}}{\Gamma_{(q)}(k(r_1-r_2)+r_1+1)} \\
&= \epsilon t^{r_1}\mathbf{E}^q_{r_1-r_2,r_1+1}(\alpha,t),
\end{aligned}$$

and the proof is completed. □

In view of Definition 1, the inequality (21) guarantees that the linear q-difference Equation (3) of the Caputo-like type is Ulam–Hyers stable with the corresponding stability constant

$$K = T^{r_1}\mathbf{E}^q_{r_1-r_2,r_1+1}(\alpha,T),$$

so that $0 \le t \le T$.

Remark 1. *We know that if* $r_2 = 0$, *then* $({}^c\mathbb{D}^{r_2}_{(q)}y)(t) = y(t)$. *In this case, the linear* q*-difference equation*

$$({}^c\mathbb{D}^{r_1}_{(q)}y)(t) - \alpha({}^c\mathbb{D}^{r_2}_{(q)}y)(t) = h(t),$$

is equivalent to the linear q*-difference equation*

$$({}^c\mathbb{D}^{r_1}_{(q)}y)(t) - \alpha y(t) = h(t),$$

and so the stability constant $\mathtt{t}^{\mathtt{r}_1}\mathbf{E}^{\mathtt{q}}_{\mathtt{r}_1-\mathtt{r}_2,\mathtt{r}_1+1}(\alpha,\mathtt{t})$ *is equal to* $\mathtt{t}^{\mathtt{r}_1}\mathbf{E}^{\mathtt{q}}_{\mathtt{r}_1,\mathtt{r}_1+1}(\alpha,\mathtt{t})$*. Hence, we can find that if* $\mathtt{r}_2 = 0$*, Theorem 4 is a generalization of Theorem 3.*

4. Stability of the Nonlinear q-Cauchy IVP

The present section deals with some qualitative properties of the nonlinear q-difference Cauchy problem of the Caputo-like type given by (4). For the existence property, we first list some needed conditions on the nonlinear real-valued function φ defined on $[0,\mathtt{T}]\times\mathbb{R}$.

$[C1]$: $\varphi : [0,\mathtt{T}]\times\mathbb{R}\to\mathbb{R}$ is continuous.
$[C2]$: $\exists\, A_\varphi > 0$ s.t. $\left|\varphi(\mathtt{t},\mathtt{y}_1)-\varphi(\mathtt{t},\mathtt{y}_2)\right| \le A_\varphi|\mathtt{y}_1-\mathtt{y}_2|,\ \forall\,\mathtt{y}_1,\mathtt{y}_2\in\mathbb{R},\mathtt{t}\in[0,\mathtt{T}]$.
$[C3]$: $\exists\, B_\varphi > 0$ s.t. $\left|\varphi(\mathtt{t},\mathtt{y})\right| \le B_\varphi(1+|\mathtt{y}|),\ \forall\,\mathtt{y}\in\mathbb{R},\mathtt{t}\in[0,\mathtt{T}]$.

Based on the above conditions, we prove a theorem about the existence property.

Theorem 5. *Let* $0<\mathtt{r}<1$, $0<\mathtt{q}<1$ *and the conditions* $[C1]$ *and* $[C2]$ *be satisfied. If*

$$A_\varphi\frac{\mathtt{T}^{\mathtt{r}}}{\Gamma_{(\mathtt{q})}(\mathtt{r}+1)} < 1,$$

then the nonlinear q*-difference Cauchy IVP (4) admits one and only one solution on* $[0,\mathtt{T}]$.

Proof. We define $H : \mathcal{C}([0,\mathtt{T}],\mathbb{R})\to\mathcal{C}([0,\mathtt{T}],\mathbb{R})$ by

$$(H\mathtt{y})(\mathtt{t}) := \mathtt{y}_0 + \frac{1}{\Gamma_{(\mathtt{q})}(\mathtt{r})}\int_0^{\mathtt{t}}(\mathtt{t}-\mathtt{q}\mathtt{w})_{(\mathtt{q})}^{(\mathtt{r}-1)}\varphi(\mathtt{w},\mathtt{y}(\mathtt{w}))\,\mathrm{d}_{(\mathtt{q})}\mathtt{w},\quad \mathtt{t}\in[0,\mathtt{T}]. \tag{27}$$

Due to the condition $[C1]$, our definition for H is well defined. Using the condition $[C2]$, for each $\mathtt{y}_1,\mathtt{y}_2\in\mathcal{C}([0,\mathtt{T}],\mathbb{R})$ and $\mathtt{t}\in[0,\mathtt{T}]$, we have

$$\begin{aligned}\left|(H\mathtt{y}_1)(\mathtt{t})-(H\mathtt{y}_2)(\mathtt{t})\right| &\le \frac{1}{\Gamma_{(\mathtt{q})}(\mathtt{r})}\int_0^{\mathtt{t}}(\mathtt{t}-\mathtt{q}\mathtt{w})_{(\mathtt{q})}^{(\mathtt{r}-1)}\left|\varphi(\mathtt{w},\mathtt{y}_1(\mathtt{w}))-\varphi(\mathtt{w},\mathtt{y}_2(\mathtt{w}))\right|\mathrm{d}_{(\mathtt{q})}\mathtt{w}\\ &\le A_\varphi\frac{1}{\Gamma_{(\mathtt{q})}(\mathtt{r})}\int_0^{\mathtt{t}}(\mathtt{t}-\mathtt{q}\mathtt{w})_{(\mathtt{q})}^{(\mathtt{r}-1)}\left|\mathtt{y}_1(\mathtt{w})-\mathtt{y}_2(\mathtt{w})\right|\mathrm{d}_{(\mathtt{q})}\mathtt{w}\\ &\le A_\varphi\frac{\mathtt{t}^{\mathtt{r}}}{\Gamma_{(\mathtt{q})}(\mathtt{r}+1)}\|\mathtt{y}_1-\mathtt{y}_2\|\\ &\le A_\varphi\frac{\mathtt{T}^{\mathtt{r}}}{\Gamma_{(\mathtt{q})}(\mathtt{r}+1)}\|\mathtt{y}_1-\mathtt{y}_2\|,\end{aligned}$$

which gives

$$\|H\mathtt{y}_1-H\mathtt{y}_2\| \le A_\varphi\frac{\mathtt{T}^{\mathtt{r}}}{\Gamma_{(\mathtt{q})}(\mathtt{r}+1)}\|\mathtt{y}_1-\mathtt{y}_2\|.$$

It is known that H is a contraction, since $A_\varphi\frac{\mathtt{T}^{\mathtt{r}}}{\Gamma_{(\mathtt{q})}(\mathtt{r}+1)} < 1$. The Banach contraction principle implies that H admits a unique fixed point, and so the q-difference Cauchy IVP (4) has one and only one solution on $[0,\mathtt{T}]$, and this ends the proof. □

In the next theorem, Schaefer's fixed point theorem implies the existence criterion for the solutions of (4).

Theorem 6. *Let* $0<\mathtt{r}<1$, $0<\mathtt{q}<1$ *and the conditions* $[C1]$ *and* $[C3]$ *be satisfied. Then, the nonlinear* q*-difference Cauchy IVP (4) admits at least one solution on* $[0,\mathtt{T}]$.

Proof. As in (27), we consider the operator H on $\mathcal{C}([0,\mathtt{T}],\mathbb{R})$. We continue the proof in several steps.

Step A: Continuity of H.

To prove the above claim, in this step, we assume the sequence $\{y_n\} \subseteq \mathcal{C}([0,T],\mathbb{R})$ so that $y_n \to y$. For each $t \in [0,T]$, we have

$$\begin{aligned}
\big|(Hy_n)(t) - (Hy)(t)\big| &= \bigg|\frac{1}{\Gamma_{(q)}(r)}\int_0^t (t-qw)_{(q)}^{(r-1)}\varphi(w,y_n(w))\,d_{(q)}w \\
&\quad - \frac{1}{\Gamma_{(q)}(r)}\int_0^t (t-qw)_{(q)}^{(r-1)}\varphi(w,y(w))\,d_{(q)}w\bigg| \\
&\leq \frac{1}{\Gamma_{(q)}(r)}\int_0^t (t-qw)_{(q)}^{(r-1)}\big|\varphi(w,y_n(w)) - \varphi(w,y(w))\big|\,d_{(q)}w \\
&\leq \|\varphi(\cdot,y_n) - \varphi(\cdot,y)\|\frac{1}{\Gamma_{(q)}(r)}\int_0^t (t-qw)_{(q)}^{(r-1)} d_{(q)}w \\
&\leq \frac{T^r}{\Gamma_{(q)}(r+1)}\|\varphi(\cdot,y_n) - \varphi(\cdot,y)\|.
\end{aligned}$$

Since φ is continuous, $\|\varphi(\cdot,y_n) - \varphi(\cdot,y)\| \to 0$ as $n \to \infty$, and so H is continuous.

Step B: H maps the bounded sets into the bounded sets in $\mathcal{C}([0,T],\mathbb{R})$.

The main purpose of this step is to show that for every $\rho > 0$, there is some $M > 0$ so that, for each function $y \in \overline{\mathbf{B}}_\rho := \{y \in \mathcal{C}([0,T],\mathbb{R}) : \|y\| \leq \rho\}$, we have

$$\|Hy\| \leq M.$$

Thus, we receive help from the condition $[C3]$, and, for each $t \in [0,T]$, we can write

$$\begin{aligned}
\big|(Hy)(t)\big| &\leq |y_0| + \frac{1}{\Gamma_{(q)}(r)}\int_0^t (t-qw)_{(q)}^{(r-1)}\big|\varphi(w,y(w))\big|\,d_{(q)}w \\
&\leq |y_0| + \frac{B_\varphi(1+\|y\|)}{\Gamma_{(q)}(r)}\int_0^t (t-qw)_{(q)}^{(r-1)} d_{(q)}w \\
&\leq |y_0| + \frac{B_\varphi(1+\rho)}{\Gamma_{(q)}(r+1)}T^r.
\end{aligned}$$

Hence,

$$\|Hy\| \leq |y_0| + \frac{B_\varphi(1+\rho)}{\Gamma_{(q)}(r+1)}T^r := M.$$

Step C: H maps the bounded sets into the equicontinuous sets in $\mathcal{C}([0,T],\mathbb{R})$.

Through some existing hypotheses, for each $t_1, t_2 \in [0,T]$ with $0 \leq t_1 < t_2 \leq T$ and $y \in \overline{\mathbf{B}}_\rho$, and from $[C3]$, we estimate

$$\begin{aligned}
\big|(Hy)(t_1) - (Hy)(t_2)\big| &= \bigg|\frac{1}{\Gamma_{(q)}(r)}\int_0^{t_1} (t_1-qw)_{(q)}^{(r-1)}\varphi(w,y(w))\,d_{(q)}w \\
&\quad - \frac{1}{\Gamma_{(q)}(r)}\int_0^{t_2} (t_2-qw)_{(q)}^{(r-1)}\varphi(w,y(w))\,d_{(q)}w\bigg| \\
&\leq \frac{B_\varphi(1+\rho)}{\Gamma_{(q)}(r)}\left(\int_0^{t_1}\left[(t_1-qw)_{(q)}^{(r-1)} - (t_2-qw)_{(q)}^{(r-1)}\right]d_{(q)}w\right. \\
&\quad \left. + \int_{t_1}^{t_2}(t_2-qw)_{(q)}^{(r-1)}\,d_{(q)}w\right)
\end{aligned}$$

$$\leq \frac{B_\varphi(1+\rho)}{\Gamma_{(q)}(r)}\left(t_1^r - t_2^r - 2(t_2-t_1)^r\right). \tag{28}$$

Since, when $t_1 \to t_2$, the R.H.S. of (28) tends to zero, H is equicontinuous.

Steps A, B, and C, along with the Arzelá-Ascoli theorem, imply the complete continuity of the operator H.

Step D: The a priori bounds.

We create a new set as $X(H) := \{y \in \mathcal{C}([0,T],\mathbb{R}) : y = \ell H y \text{ for some } \ell \in (0,1)\}$ and prove that it is bounded. Assume $y \in X(H)$. Clearly, for some $\ell \in (0,1)$, we have $y = \ell H y$. Now, for all $t \in [0,T]$, we have

$$\begin{aligned}
|y(t)| &\leq |(Hy)(t)| \\
&\leq y_0 + \frac{1}{\Gamma_{(q)}(r)}\int_0^t (t-qw)_{(q)}^{(r-1)}\left|\varphi(w,y(w))\right| d_{(q)}w \\
&\leq y_0 + \frac{B_\varphi}{\Gamma_{(q)}(r)}\int_0^t (t-qw)_{(q)}^{(r-1)}\left|1+y(w)\right| d_{(q)}w \\
&\leq y_0 + \frac{B_\varphi}{\Gamma_{(q)}(r)}\int_0^t (t-qw)_{(q)}^{(r-1)} d_{(q)}w \\
&+ \frac{B_\varphi}{\Gamma_{(q)}(r)}\int_0^t (t-qw)_{(q)}^{(r-1)}\left|y(w)\right| d_{(q)}w \\
&\leq y_0 + \frac{B_\varphi}{\Gamma_{(q)}(r+1)}t^r + \frac{B_\varphi}{\Gamma_{(q)}(r)}\int_0^t (t-qw)_{(q)}^{(r-1)}\left|y(w)\right| d_{(q)}w.
\end{aligned}$$

If we use the Gronwall's inequality in the quantum sense (Theorem 1), then we have

$$\begin{aligned}
|y(t)| &\leq \left(y_0 + \frac{B_\varphi}{\Gamma_{(q)}(r+1)}t^r\right)\mathbf{E}_r^q\left(\alpha, \frac{B_\varphi}{r}t^r\right) \\
&\leq \left(y_0 + \frac{B_\varphi}{\Gamma_{(q)}(r+1)}T^r\right)\mathbf{E}_r^q\left(\alpha, \frac{B_\varphi}{r}T^r\right) < \infty.
\end{aligned}$$

This proves the boundedness of the set $X(H)$.

Schaefer's fixed point theorem guarantees that H has a fixed point and it is the solution of the q-difference nonlinear Cauchy IVP (4). The proof is complete now. □

To complete our stability analysis, consider the inequality

$$\left|({}^{c}\mathbb{D}_{(q)}^{r}y)(t) - \varphi(t,y(t))\right| \leq \Lambda(t), \tag{29}$$

for $t \in [0,T]$.

Another condition is needed as follows.

$[C4]$: Assume that $\Lambda \in \mathcal{C}([0,T],\mathbb{R}^+)$ is increasing. There is some $C_\Lambda > 0$ such that

$$\int_0^t (t-qw)_{(q)}^{(r-1)}\Lambda(w)\, d_{(q)}w \leq C_\Lambda \Lambda(t), \quad t \in [0,T]. \tag{30}$$

The following stability criterion can be proven for the q-difference nonlinear Cauchy IVP (4) now.

Theorem 7. *Let* $[C1]$, $[C2]$ *and* $[C4]$ *be satisfied. If*

$$A_\varphi \frac{T^r}{\Gamma_{(q)}(r+1)} < 1,$$

then the q*-difference nonlinear Cauchy IVP* (4) *is generalized Ulam–Hyers–Rassias stable with respect to the function* Λ *on* $[0, \mathtt{T}]$.

Proof. We assume that $y \in \mathcal{C}([0,\mathtt{T}],\mathbb{R})$ is a solution of the q-difference Cauchy IVP (4). The conclusion of Theorem 5 guarantees that

$$\hbar(\mathtt{t}) = y_0 + \frac{1}{\Gamma_{(q)}(\mathtt{r})}\int_0^{\mathtt{t}} (\mathtt{t}-q\mathtt{w})_{(q)}^{(\mathtt{r}-1)} \varphi(\mathtt{w},\hbar(\mathtt{w}))\, d_{(q)}\mathtt{w},$$

is the unique solution of the IVP

$$\begin{cases} ({}^{c}\mathbb{D}^{\mathtt{r}}_{(q)}\hbar)(\mathtt{t}) = \phi(\mathtt{t},\hbar(\mathtt{t})), \\ \hbar(0) = y_0, \quad \mathtt{t} \in [0,\mathtt{T}], \quad 0 < \mathtt{r} < 1. \end{cases} \tag{31}$$

By integrating the inequality (29) on $[0, \mathtt{t}]$ and by $[C4]$, we have

$$\begin{aligned} &\left| y(\mathtt{t}) - y_0 - \frac{1}{\Gamma_{(q)}(\mathtt{r})}\int_0^{\mathtt{t}} (\mathtt{t}-q\mathtt{w})_{(q)}^{(\mathtt{r}-1)} \varphi(\mathtt{w},y(\mathtt{w}))\, d_{(q)}\mathtt{w} \right| \\ &\qquad \le \frac{1}{\Gamma_{(q)}(\mathtt{r})}\int_0^{\mathtt{t}} (\mathtt{t}-q\mathtt{w})_{(q)}^{(\mathtt{r}-1)} \Lambda(\mathtt{w})\, d_{(q)}\mathtt{w} \\ &\qquad \le \frac{1}{\Gamma_{(q)}(\mathtt{r})} C_\Lambda \Lambda(\mathtt{t}). \end{aligned}$$

Hence, we have

$$\begin{aligned} \left| y(\mathtt{t}) - \hbar(\mathtt{t}) \right| &= \left| y(\mathtt{t}) - y_0 - \frac{1}{\Gamma_{(q)}(\mathtt{r})}\int_0^{\mathtt{t}} (\mathtt{t}-q\mathtt{w})_{(q)}^{(\mathtt{r}-1)} \varphi(\mathtt{w},\hbar(\mathtt{w}))\, d_{(q)}\mathtt{w} \right| \\ &\le \left| y(\mathtt{t}) - y_0 - \frac{1}{\Gamma_{(q)}(\mathtt{r})}\int_0^{\mathtt{t}} (\mathtt{t}-q\mathtt{w})_{(q)}^{(\mathtt{r}-1)} \varphi(\mathtt{w},y(\mathtt{w}))\, d_{(q)}\mathtt{w} \right. \\ &+ \frac{1}{\Gamma_{(q)}(\mathtt{r})}\int_0^{\mathtt{t}} (\mathtt{t}-q\mathtt{w})_{(q)}^{(\mathtt{r}-1)} \varphi(\mathtt{w},y(\mathtt{w}))\, d_{(q)}\mathtt{w} \\ &\left. - \frac{1}{\Gamma_{(q)}(\mathtt{r})}\int_0^{\mathtt{t}} (\mathtt{t}-q\mathtt{w})_{(q)}^{(\mathtt{r}-1)} \varphi(\mathtt{w},\hbar(\mathtt{w}))\, d_{(q)}\mathtt{w} \right| \\ &\le \left| y(\mathtt{t}) - y_0 - \frac{1}{\Gamma_{(q)}(\mathtt{r})}\int_0^{\mathtt{t}} (\mathtt{t}-q\mathtt{w})_{(q)}^{(\mathtt{r}-1)} \varphi(\mathtt{w},y(\mathtt{w}))\, d_{(q)}\mathtt{w} \right| \\ &+ \frac{1}{\Gamma_{(q)}(\mathtt{r})}\int_0^{\mathtt{t}} (\mathtt{t}-q\mathtt{w})_{(q)}^{(\mathtt{r}-1)} \left| \varphi(\mathtt{w},y(\mathtt{w})) - \varphi(\mathtt{w},\hbar(\mathtt{w})) \right| d_{(q)}\mathtt{w} \\ &\le \frac{1}{\Gamma_{(q)}(\mathtt{r})} C_\Lambda \Lambda(\mathtt{t}) + A_\varphi \frac{1}{\Gamma_{(q)}(\mathtt{r})}\int_0^{\mathtt{t}} (\mathtt{t}-q\mathtt{w})_{(q)}^{(\mathtt{r}-1)} \left| y(\mathtt{w}) - \hbar(\mathtt{w}) \right| d_{(q)}\mathtt{w}. \end{aligned}$$

Now, the q-Gronwall inequality (Theorem 1) implies that

$$\begin{aligned} \left| y(\mathtt{t}) - \hbar(\mathtt{t}) \right| &\le \frac{C_\Lambda}{\Gamma_{(q)}(\mathtt{r})} \Lambda(\mathtt{t}) \mathbf{E}^{q}_{\mathtt{r}}(\alpha, \frac{A_\varphi}{\mathtt{r}} \mathtt{t}^{\mathtt{r}}) \\ &\le \frac{C_\Lambda}{\Gamma_{(q)}(\mathtt{r})} \mathbf{E}^{q}_{\mathtt{r}}(\alpha, \frac{A_\varphi}{\mathtt{r}} \mathtt{T}^{\mathtt{r}}) \Lambda(\mathtt{t}). \end{aligned}$$

Let $K := \frac{C_\Lambda}{\Gamma_{(\mathtt{q})}(\mathtt{r})}\mathbf{E}^{\mathtt{q}}_{\mathtt{r}}(\alpha, \frac{A_\varphi}{\mathtt{r}}\mathtt{T}^{\mathtt{r}})$. Then,

$$\left|\mathtt{y}(\mathtt{t}) - \hbar(\mathtt{t})\right| \leq K\Lambda(\mathtt{t}), \quad \mathtt{t} \in [0, \mathtt{T}].$$

By Definition 2, we find that the q-difference Cauchy IVP (4) is generalized Hyers–Ulam–Rassias stable with respect to the function Λ on $[0, \mathtt{T}]$, and this completes the proof. □

5. Example

An example of the q-Cauchy IVP is given in this section to check the applicability of the main results.

Example 1. *Let us consider a* q*-difference nonlinear IVP of the Caputo-like type*

$$\begin{cases} ({}^{c}\mathbb{D}^{0.75}_{(0.5)}\mathtt{y})(\mathtt{t}) = \dfrac{|\sin \mathtt{y}(\mathtt{t})| \cos \mathtt{t}}{6 + 6|\sin \mathtt{y}(\mathtt{t})|}, \\ \mathtt{y}(0) = 10, \end{cases} \tag{32}$$

and consider an inequality given by

$$\left|({}^{c}\mathbb{D}^{0.75}_{(0.5)}\mathtt{y})(\mathtt{t}) - \frac{|\sin \mathtt{y}(\mathtt{t})| \cos \mathtt{t}}{6 + 6|\sin \mathtt{y}(\mathtt{t})|}\right| \leq \Lambda(\mathtt{t}),$$

for each $\mathtt{t} \in [0, 5]$. In fact, we set $\mathtt{r} = 0.75$, $\mathtt{q} = 0.5$, $\mathtt{T} = 5$, and $\mathtt{y}_0 = 10$. Also, for each $(\mathtt{t}, \mathtt{y}) \in [0, 5] \times \mathbb{R}$, define

$$\varphi(\mathtt{t}, \mathtt{y}) = \frac{|\sin \mathtt{y}| \cos \mathtt{t}}{6(1 + |\sin \mathtt{y}|)}.$$

For each $\mathtt{y}_1, \mathtt{y}_2 \in \mathbb{R}$, we have $A_\varphi = \frac{1}{6}$, since

$$\left|\varphi(\mathtt{t}, \mathtt{y}_1(\mathtt{t})) - \varphi(\mathtt{t}, \mathtt{y}_2(\mathtt{t}))\right| \leq \frac{\cos \mathtt{t}}{6}\left|\mathtt{y}_1(\mathtt{t}) - \mathtt{y}_2(\mathtt{t})\right| \leq \frac{1}{6}\left|\mathtt{y}_1(\mathtt{t}) - \mathtt{y}_2(\mathtt{t})\right|.$$

Also, we have

$$A_\varphi \frac{\mathtt{T}^{\mathtt{r}}}{\Gamma_{(\mathtt{q})}(\mathtt{r} + 1)} = \frac{1}{6}\frac{5^{0.75}}{\Gamma_{(0.5)}(1.75)} = 0.589170195 < 1.$$

Set $\Lambda(\mathtt{t}) = \exp(\mathtt{t})$ for $\mathtt{t} \in [0, 5]$ and $C_\Lambda = \frac{20}{3} > 0$.

Notice that

$$\begin{aligned} \int_0^{\mathtt{t}} (\mathtt{t} - \mathtt{q}\mathtt{w})^{(\mathtt{r}-1)}_{(\mathtt{q})}\Lambda(\mathtt{w})\, d_{(\mathtt{q})}\mathtt{w} &= \int_0^{\mathtt{t}} (\mathtt{t} - 0.5\mathtt{w})^{(-1/4)}_{(0.5)} \exp(\mathtt{w})(\mathtt{w})\, d_{(\mathtt{q})}\mathtt{w} \\ &\leq \frac{4}{3}\mathtt{t} \exp(\mathtt{t}) \leq C_\Lambda \Lambda(\mathtt{t}), \end{aligned}$$

for all $\mathtt{t} \in [0, 5]$. Since all conditions given in Theorems 5 and 7 are satisfied, the unique solution of the q-difference IVP (32) is generalized Ulam–Hyers–Rassias stable with respect to the function $\Lambda(\mathtt{t}) = \exp(\mathtt{t})$.

6. Conclusions

In this study, for some linear and nonlinear fractional q-difference equations, a stability analysis was conducted in the Ulam–Hyers and Ulam–Hyers–Rassias sense. Our main tool for this purpose was the quantum Laplace transform and quantum Mittag–Leffler function. Moreover, we considered the nonlinear Cauchy q-difference initial value problem and proved the existence results for its solution. Next, we used the quantum Laplace transform and the q-Gronwall inequality to prove the generalized Ulam–Hyers–Rassias stability for the given Cauchy q-difference problem. In future studies, we can also use this technique for every nonlinear q-difference equation due to every real-world phenomena and analyze its numerical and dynamical behaviors based on the small changes in the initial data. Moreover, if we can define the (p, q)-Mittag–Leffler function, then one can extend these results by using the (p, q)-Laplace transform in future research projects.

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