Advantages of the Complex Taylor Series Expansion Method for determining Circuit Outputs

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Abstract

The complex Taylor series expansion (CTSE) is a method to compute derivatives numerically with machine precision for real valued functions. This method converts a real-valued function to complex-valued and introduces a small imaginary step to the parameter of interest. In this work, CTSE had been used in an electrical circuit to find the current flowing through a capacitor with respect to the voltage across the capacitor, where the voltage across the capacitor is a complicated solution of a differential equation. The advantage of CTSE is that it can be used to find accurate derivatives of arbitrary functions. A limitation of CTSE is that it can only be used to compute the first order derivative of a function. A numerical example is presente to demonstrate the accuracy of the method.

Key Words: Electrical Circuit, Complex Taylors Series Expansion, Sensitivities

Introduction

Circuits have an important role in electrical engineering applications, especially in analysis. In choosing a method for computing and simulating circuit behaviors, one is concerned with accuracy and the ease of use. A fundamental set of methods used for differentiating functions is the finite difference methods (FDM). FDM's are particulary easy to implement, but lack accuracy due to the dilemma of using a small step size h to minimize the truncation error vs. avoiding a small h because of the subtractive cancellation error [6]. To counter this dilemma, a relatively new method is the complex Taylor series expansion method which uses the imaginary axis to bypass the step size issue prevalent with the finite difference methods.

CTSE has a large list of applications that vary widely in other discipline and fields of study. For example, in mechanical engineering the energy release rates of strain energy with respect to crack pertubations can be numerically calculated by extending the crack by a very small quantity along the imaginary direction of the complex coordinate system. This results in a complex valued solution for the strain energy function with the imaginary component containing the energy release rate [5]. In chemical engineering, atmospheric chemistry-transport models for understanding emmision effects on the atmosphere were developed by applying a complex value to a multivalued function and obtaining an equation to compute the impact of particulate [2].

The objective of this work is to show that the CTSE method can be applied to an electrical engineering application by introducing the method to simulate a simple circuit described by a first order differential equation, and comparing the results with that of the finite difference method as proof of superior precision.

Finite Difference Methods (FDM)

The Taylor series is an infinite series which provides a means to predict a function about a point, x, in terms of the function value and it's derivatives at another point, a. Taylor's theorem states that any smooth, real or complex, function can be expressed as a Taylor series given by the polynomial [1]

$$f(x) = f(a)+f'(a)(x-a)+\frac{1}{2}f''(a)(x-a)^2+\frac{1}{6}f'''(a)(x-a)^3+...+\frac{1}{n!}f^{(n)}(a)(x-a)^n \quad (1)$$

where $f(x)$ denotes the real valued function, $n!$ denotes the facotrial of n and $f^{(n)}(a)$ denotes the nth derivative of f evaluated at a.

Derived from equation 1, the difference method is one of the simplest and most used forms of approximating derivatives for real valued functions. By adding a small step size h, to x and solving for the first derivative yields the following equation

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} + O(h) \quad (2)$$

For more details on the derivation of the formulas and implementations of these methods, see reference [1].

Equation 2 is the forward difference method, because it utilizes data at $x+h$ and x to estimate the derivative. One drawback from using the Taylor series expansion is that the truncation error is of $O(h)$ in equation 2. Truncation errors are those that result from using an approximation in place of an exact mathematical procedure [1]. In these cases, the truncation comes about due to approximating the infinite Taylor series expansion using only the first few terms.

Another way to approximate the first derivative through the Taylor series expansion is with the central difference method which takes data values around x at $x+h$ and $x-h$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x-h)}{2h} + O(h^2) \tag{3}$$

In equation 3, the truncation error is $O(h^2)$ giving the central difference a better approximation when compared to the forward difference as h becomes smaller and approaches zero. For example, if h is halved in the forward difference method, the truncation error would also be halved, whereas in the central difference method the truncation error would be quartered for the central difference.

Complex Taylor Series Expansion (CTSE)

The complex Taylor series expansion (CTSE) approximation is an equally simple first derivative estimation for real functions and may be obtained using complex calculus. If $f(x)$ is a real analytic function of real variables, one can expand it in a Taylor series about a real point x using an imaginary step ih [4].

$$f(z) = f(a) + f'(a)(z-a) + \frac{1}{2}f''(a)(z-a)^2 + \cdots + \frac{1}{n!}f^{(n)}(a)(z-a)^n$$

Evaluating the series at the point $a = x \in \mathbb{R}$, and assuming that our complex number $z = x + ih$.

$$\begin{aligned} f(x+i\,h) &= f(x) + f'(x)(ih) + \frac{f''(x)}{2}(ih)^2 + \frac{f'''(x)}{6}(ih)^3 + \cdots \\ &= f(x) + ihf'(x) - \frac{h^2}{2}f''(x) - \frac{ih^3}{6}f'''(x) + \cdots \end{aligned}$$

After taking the imaginary part of the function and dividing by h we have

$$\frac{\operatorname{Im}[f(x+ih)]}{h} = f'(x) - \frac{h^2}{6}f'''(x) = f'(x) - O(h^2)$$

Solving this series for the first derivative with the complex value z yields the following

$$f'(x) = \lim_{h \to 0} \frac{\text{Im}\left[f(x+ih)\right]}{h} + O(h^2) \tag{4}$$

Similar to the central difference method, the CTSE method includes a truncation error of the $O(h^2)$. One major advantage that the CTSE method has over finite different methods is that the derivative approximation does not include any subtractive cancellation. The subtractive cancellation of the finite difference methods will be shown to be a flaw in estimating derivatives of $f(x)$ as h begins to approach values closer to zero while CTSE will continue to provide very precise results.

With this superior approximation for numerical differentiation in mind, we will implement CTSE for a simple circuit problem involving an AC signal source, a resistor and a capacitor with the hypothesis that the current can be numerically calculated in the time domain precisely using CTSE. This work will also show in later sections that the CTSE method gives more accurate results with respect to the exact current than the central difference method.

Materials and Methods

Consider the application of the complex Taylor series expansion to the following circuit:

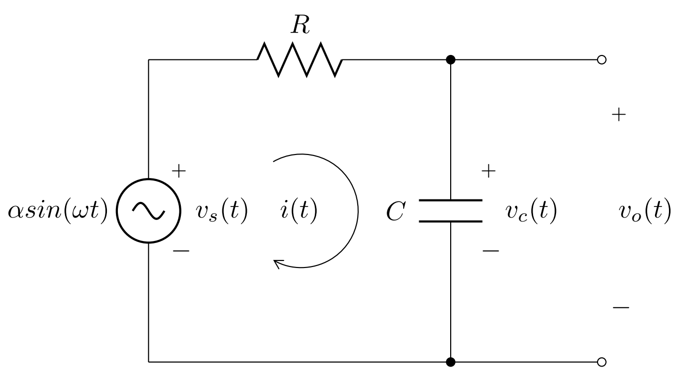

Figure 1: RC Circuit

The RC circuit above, which includes a resistor and capacitor, was chosen for the reason that the current flowing through the capacitor is equivalent to the derivative of the voltage multiplied by the time constant of the system. Alternatively the RL circuit, which includes a resistor and inductor, could have been used as well because of a similar relation between the current and voltage.

The KVL (Kirchoff's Voltage Law) is a fundamental tool for electrical engineering which states that "the algebraic sum of the voltages around any loop in a circuit is identically zero for all time [3]." Thus, taking the KVL of figure 1 in the direction of $i(t)$ yields

$$v_s(t) - i(t)R - v_c(t) = 0 \tag{5}$$

where $v_s(t)$ is the source voltage, $i(t)$ is the total current in the system, and $v_c(t)$ is the voltage across the capacitor.

After rearranging equation 5 and substituting, $i(t) = i_c(t) = C\frac{d}{dt}v_c(t)$ we obtain

$$CR\frac{d}{dt}v_c(t) + v_c(t) = v_s(t)$$

Substituing and dividing by $\tau = CR$, the equation yields

$$\frac{d}{dt}v_c(t) + \frac{1}{\tau}v_c(t) = \frac{1}{\tau}v_s(t) \tag{6}$$

where τ is the time constant of the circuit and $i(t) = i_c(t)$ due to the fact that the circuit is connected in series.

Solving the first-order differential equation in equation 6 given the initial value $v_c(t = 0) = 0$ yields

$$v_c(t) = \left(\frac{\alpha}{1 + (\omega\tau)^2}\right)\left(\omega\tau e^{-t/\tau} - \omega\tau\cos(\omega t) + \sin(\omega t)\right) \tag{7}$$

As can be seen, the voltage function of the capacitor is lengthy and taking the analytical derivative may be prone to error, therefore using CTSE in a programming language precisely approximates the current in the system without the need of having to analytically compute the derivative and multiplying by C. For comparison purposes, equation 8 shows the exact current of Figure 1.

$$\begin{aligned} i_c(t) &= C\frac{d}{dt}v_c(t) \\ &= C\left(\frac{\alpha}{1 + (\omega\tau)^2}\right)\left(-\omega e^{-t/\tau} + \omega^2\tau\cos(\omega t) + \omega\sin(\omega t)\right) \end{aligned} \tag{8}$$

Results and Discussion

Figure 2 shows the relative error in computing $\frac{d}{dx}e^x$ as h begins to draw closer to zero using three different methods: the forward difference method, the central difference method, and the complex Taylor series expansion.

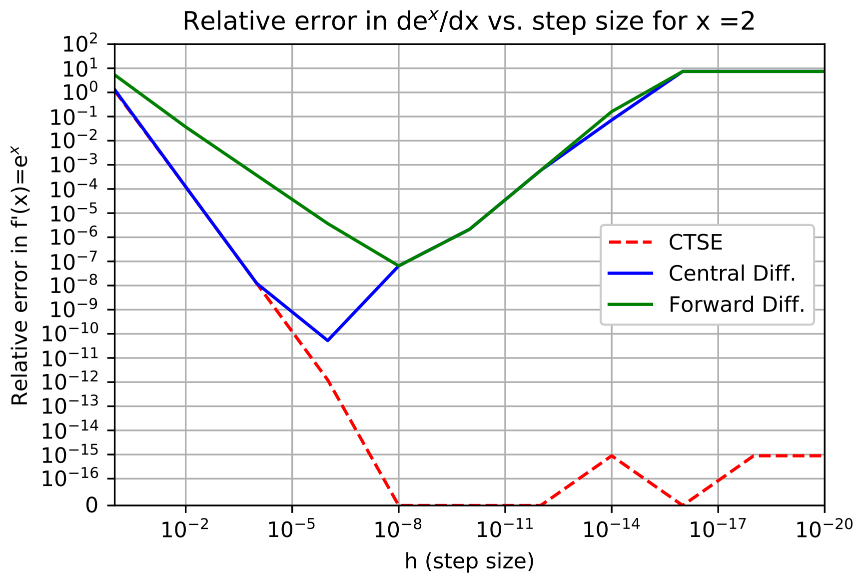

Figure 2: Sensitivty of CTSE and Finite difference methods as h reduces to 0

As h is reduced, the truncation error in the forward difference method is reduced, thereby reducing the overall error. After h reaches the specific size 10^{-8}, the subtractive cancellation error occurs and causes the overall absolute error in the method to increase. The subtractive cancellation error is a direct result of the forward difference method formula that occurs for h values that are too small. When the h size is small enough, the difference on the numerator between $f(x+h)$ and $f(x)$ in equation 2 is amplified by the reciprocal of the step size h.

It can be seen in the previous figure that the slope of the central difference method is steeper and positioned lower than the forward difference method for $h \gtrsim 10^{-6}$. This is due to the central difference method's truncation error being $O(h^2)$ in comparison to the forward difference method's truncation error being $O(h)$. As h reduces to values lower than 10^{-6}, the subtractive cancellation error occurs for the central difference method due to the difference between $f(x+h)$ and $f(x-h)$ on the numerator in equation 3, similar to the forward difference method.

Similar to the finite difference methods, the CTSE method is derived from the Taylor series expansion, so the truncation error exists. Additionally, its convergence is $O(h^2)$, which explains why in Figure 2 the CTSE and the central difference are on top of each other for a short duration; however, unlike the two finite difference methods, the CTSE method does not have a subtractive component in its numerator; therefore, the subtractive cancellation error does not exist. With no subtractive error, CTSE continues to converge for even smaller step sizes, and eventually leads to the error becoming independent of h. For these reasons, it is clear that CTSE has better precision.

Values for the given components in Figure 1 were chosen to be $C = 1\mu$F, $R = 10k\Omega$, and $v_s = 4\sin(100t)$ with and imaginary step size of $h = 10^{-15}$. Any step size for $h \leq 10^{-8}$ would be sufficiently precise as seen in Figure 2. Tables 1 and 2 show and compare the exact current $i_c(t)$ with the numerical current values using the CTSE method and the central difference method.

In Table 1, the analytical current and the CTSE result are exact. The reason being that the error in the CTSE method does not occur until after the 16th decimal place. Thus, the CTSE has machine precision due to the machine epsilon (also known as unit roundoff) for a 64-bit number being precise up to the 16th decimal place (10^{-16}). Whereas, the central difference method shows errors within the first few decimal places.

In Table 2, the relative error when compared to the exact solution is shown and directly exhibits the machine precision of the CTSE method with relative error being near or lower than the machine epsilon .

t (s)	Exact $i_c(t)$ (mA)	CTSE Method (mA)	central Difference Method (mA)
0	0.0000000000000000	0.0000000000000000	-0.0000066778862220
.01	0.2027787699009188	0.2027787699009188	0.2028044399082773
.02	0.0715630614083853	0.0715630614083853	0.0715649761673376
.03	-0.1797319113816884	-0.1797319113816884	-0.1797006987658278
.04	-0.2857523510120549	-0.2857523510120549	-0.2857158953872840
.05	-0.1364000072397995	-0.1364000072397995	-0.1363353874239692

Table 1: Current values in mA units using the exact equation, the CTSE method ($h = 10^{-15}$), and the central difference ($h = 10^{-15}$) method

t (s)	Relative error between Exact and CTSE	Relative error between Exact and CDM
0	nan	inf
.01	0.0	-0.000126591197743
.02	0.0	-2.67562470721e-05
.03	-1.54427643941e-16	0.000173662070473
.04	- 0.0	0.000127577689708
.05	-2.03486613947e-16	0.000473752290326

Table 2: Relative errors in the curretn $i(t)$ for CTSE and central difference method relative to the exact current

Figure 3 is a Python generated plot showing the sinusoidal input-voltage in comparison to the voltage and current passing through the capacitor as an output.

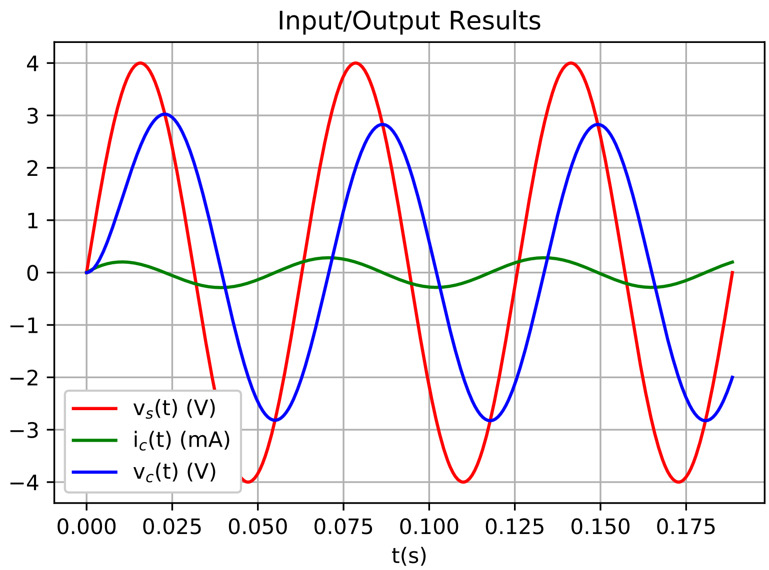

Figure 3: Simulation of Figure 1 with $C = 1\mu F$, $R = 10k\Omega$, and $v_s = 4\sin(100t)$

Conclusion

This work demonstrated that the CTSE method is a precise tool for numerically differentiating functions with machine precision and that it can be used to simulate the behavior of a circuit containing a capacitor and resistor. Due to this result with an RC circuit, RL and RLC circuits can be implemented by combining capacitors and inductors into a network therefore creating a higher order differential equation. Hence, CTSE can be implemented for complicated circuits including energy storage devices. Results concerning the convergence of the finite difference methods and the CTSE method agree with previous studies [6]. CTSE continues to converge to lower amounts of error until the error is no longer dependent on the step size, while for the finite difference methods, subtractive cancellation error takes place when the step size becomes too small. Additionally, when compared to the exact current of an RC circuit, the current using the CTSE method had more precise results when compared to the current evaluated using the central difference method.

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