ON THE ENERGY CONSERVED IN A BUCKLING FUNG HYPERELASTIC CYLINDRICAL SHELL SUBJECTED TO TORSION, INTERNAL PRESSURE AND AXIAL TENSION

by

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DEDICATION

This thesis is dedicated to my parents who have tirelessly supported me in the pursuit of my dreams. Thank you Mom for listening to me rant, comforting me, and responding to my poop emoji text messages early in the morning. I will forever cherish your wisdom, generosity, and boundless capacity to love.

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A theoretical model is proposed for the buckling of a three-dimensional vein subjected to torsion, internal pressure, and axial tension using energy conservation methods. The vein is assumed to be an anisotropic hyperelastic cylindrical shell which obeys the Fung constitutive model. Finite deformation theory for thick-walled blood vessels is used to characterize the vessel dilation in the pre-buckling state.

The pre-buckling state is identified by its midpoint and then perturbed by a displacement vector field dependent on the circumferential and axial directions to define the buckled state. The total potential energy functional of the system is extremized by minimizing the first variation with respect to the elements of the set of all continuous bounded functions on $\mathbb{R}^3$. The Euler-Lagrange equations form three coupled linear partial differential equations with Dirichlet boundary conditions characterizing the buckling displacement field under equilibrium.

A second solution method approximates the first variation of the total potential energy functional using a variational Taylor series expansion. The approximation is minimized and combined with equations of equilibrium derived from elasticity theory to yield a polynomial relating buckling eigenmodes, material parameters, geometric parameters, and the critical angle of twist which induces buckling. Various properties of the total potential energy functional specific to the problem are proved. Another solution method is outlined using the first variation approximation and the basis of the kernel of the linear transformation which maps buckling displacement amplitudes during static equilibrium.

TABLE OF CONTENTS

LIST OF FIGURES

Chapter 1: INTRODUCTION

In this study we model the mechanical failure (buckling type) of an anisotropic Fung hyperelastic cylindrical shell subjected to torsion, internal pressure, and axial tension using energy conservation methods. A shell is a vessel with a sufficiently small thickness such that there is a constant stress distribution across the vessel wall. For materials with sufficiently large thicknesses, the stresses necessarily depend on the radius thereby invalidating some assumptions made in this work. Therefore caution should be exercised when translating the methods used here to other geometries. The approach used in this work combines areas of small (infinitesimal) and large (finite) strain theories and is intended to be more of an estimation and a stepping stone for future research.

We assume that the vessel's mechanical failure state can be obtained by considering a small displacement perturbation of the stable state just before buckling. The approach is derived from first principles without any engineering-type approximations. However, when we approximate the first variation of the total potential energy functional using a Taylor series we will require the assumption from infinitesimal strain theory that each perturbation is sufficiently small so that any displacement terms raised to the power of two or greater can be neglected. This is a necessary approximation so that the strain energy density function becomes analytically integrable.

Those well-versed in finite strain theory may criticize any first-order approximations as being woefully inadequate given the Green-Lagrange strain-displacement relations necessarily include second order displacement terms, to wit $E_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i} + u_{m,i}u_{m,j})$, as opposed to the strain-displacement relations from infinitesimal theory which approximate $E_{ij} \approx \frac{1}{2}(u_{i,j} + u_{j,i})$. It is a valid criticism and as a result of experimental data analyzed in our previous work "Solutions to the First-Order Buckling Equations of a Fung Hyperelastic Cylindrical Shell Subjected to Torsion, Internal Pressure, and Axial Tension" we believe this tax is paid in the suggested theoretical buckling modes. This is in direct contrast to linear elastic materials in which first-order approximations and single term trigonometric displacement assumptions tend to accurately predict experimental buckling modes.

In computing and minimizing the first variation of the total potential energy functional we will show two approaches. In the first we will approximate the buckled Q from the Fung strain energy function to the first-order and e^Q to the second-order, and in the second we will approximate both to the second order. Variational calculus parses higher degrees of nonlinearity than traditional buckling solution methods such as deriving the equilibrium equations from elasticity theory or shell theory. However we still require some simplifications in the context of the Fung hyperelastic constitutive model so that the Euler-Lagrange equations are linear. In our case we show both approximations to illustrate the complexities that arise when increasing the accuracy of the estimation of the buckled e^Q.

Each of the solution methods presented in this work utilize the same fundamental concept. That is to say that the external forces applied to the vessel impart energy which is stored within the vessel as deformation. We assume that no energy is lost in the process, there is no heat transfer, and the loading is sufficiently slow so that there is approximately no kinetic energy in the vessel.

The most promising solution method presented here uses the calculus of variations as we are able to better approximate the true nonlinear nature of the strain energy. In the case of a first-order approximation for buckled Q the coupled linear Euler-Lagrange equations are nice and can be solved analytically and will satisfy fixed-end boundary conditions. Furthermore, the analytical solutions may speak to the properties that other buckling displacement fields share. On the other hand, the other two methods presented require a prescribed buckling displacement field assumption. For simplicity we will use buckling displacements typically assumed in linear elastic buckling theory, and it should be noted that use of the displacement field requires the accompanying assumption that the boundary conditions have no impact on the rest of the vessel due to the length of the system.

In future work we will therefore focus on improving the assumptions used in deriving the Euler-Lagrange equations. We will more closely examine the equations that arise from taking a second-order approximation for e^Q in the hopes of solving them and obtaining more accurate buckling displacements and modes. We may also focus on adjusting the reference point of the

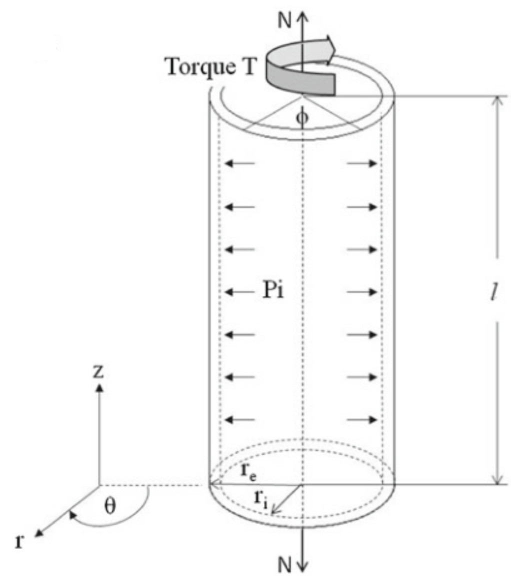

Figure 1.1: Triaxially loaded blood vessel [9]

pre-buckling vessel to the stress free configuration instead of the near-buckling configuration used here. There are some significant challenges with this approach that are outlined in our discussion of future work.

It is the aim of this work to encourage further research on theoretically modeling nonlinear buckling. If the simplifications used in deriving the Euler-Lagrange equations can be validated, that is to say that a second-order approximation is sufficient for finite deformations, then there is a great potential for analytical solutions to a wide array of nonlinear buckling problems and not necessarily just for shells. The alternative is perhaps using nonlinear shell theory and hoping that the partial differential equations are solvable.

1.1 Problem Statement

We wish to obtain a relation for the critical angle of twist of a triaxially loaded, hyperelastic, anisotropic cylindrical shell as a function of material parameters, vessel geometry, and buckling eigenmodes. Alternatively we may seek the critical torsion as a function of a prescribed angle of twist. Specifically in both cases we wish to model the torsional buckling of blood vessels. We initially consider veins since the ratio of thickness to length is smaller than that of arteries, however

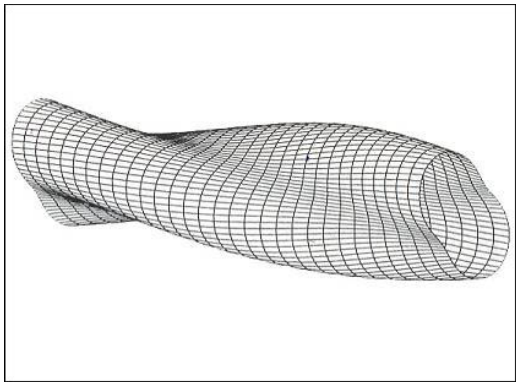

Figure 1.2: Example of linear elastic torsional buckling

by definition of what constitutes a shell we should be able to also consider arteries. The geometry of the vessel is assumed to be that of a straight, thin-wall cylinder with constant thickness (Fig 1.1).

Since veins *in vivo* are naturally subjected to axial tension and pulsatile pressure inflation we consider three loads: increasing torsion applied to both ends of the vein, passive axial tension to maintain a stretch ratio, and constant internal pressure. Once the applied torsional load reaches a critical level the vein undergoes mechanical failure and begins to collapse, or buckle. An example of a torsionally buckled, linear elastic material is shown in Figure 1.2.

The buckling patterns shown in finite-element simulations of Fung blood vessels with these loading conditions differ from that seen experimentally. In most of the experiments the vessels have some sort of defect which causes the vessels to collapse on a small region in roughly the center of the vessel [10]. The finite-element simulations indicate a more globalized nonlinear buckling unless a defect is specifically introduced into the vessel geometry. The original intention of the analytical model was to ignore the unbuckled regions of the vessel and instead assume a global mechanical failure over a length $L_0 = \lambda\Lambda L(\alpha - \beta)$ where α and β are fractions between 0 and 1, $\lambda\Lambda$ denotes the axial stretch ratio before buckling occurs, and L is the undeformed vessel length.

In our previous work we noticed that taking $\beta := 0$ and $\alpha := 1$ was sufficient in predicting the experimental critical angle of twist. This perhaps implies that a vessel with a defect buckles at a similar angle of twist as an idealized one but simply exhibits a different morphology. Considering the entire vessel length also gives more credence to our buckling displacement field assumption which necessarily requires that the ends of the vessel are sufficiently far away such that they have no impact on the vessel. In the remainder of this model we will therefore use $\beta := 0$ and $\alpha := 1$ although the notation used applies to any fractions of β and α.

1.2 Solution Outline

We can decompose the fully loaded vein into separate and sequential configuration states such that the material volume of the vein under one set of loads is the domain of a component-wise continuous vector function that maps the material volume to a new conformation which includes all previous loads and one or more new loads. Intuitively we are applying loads cumulatively one after the other rather than at the same time (Fig 1.3).

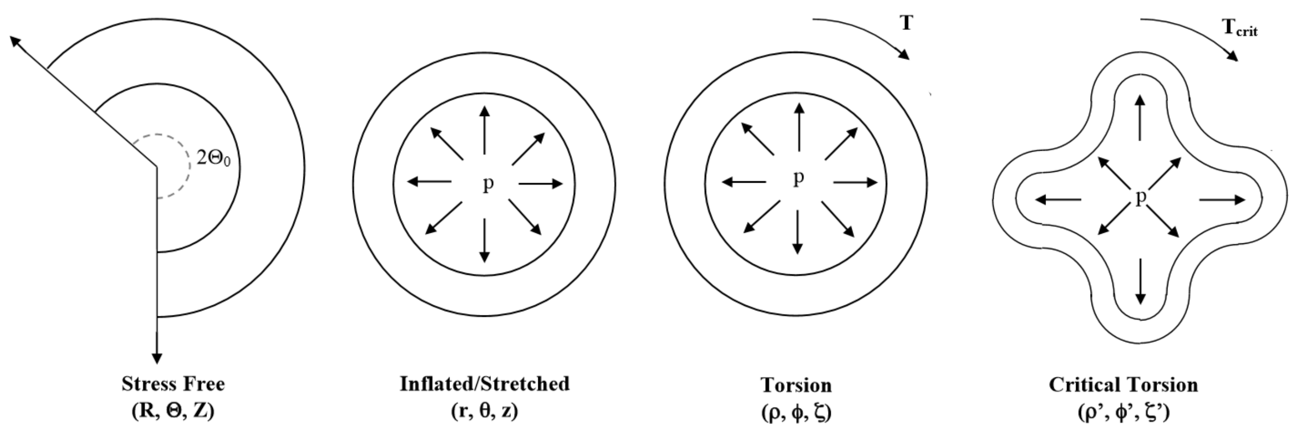

Figure 1.3: Sequential stress states of a buckled cylinder under tension, internal pressure, and torsion

$$(R, \Theta, Z) \rightarrow (r, \theta, z) \rightarrow (\rho, \phi, \zeta) \rightarrow (\rho', \phi', \zeta') \tag{1.1}$$

These states are mapped as above where the tuples represent the vein stress-free, inflated with pressure and under tension, under torsion (and inflated and stretched), and under critical torsion (and inflated and stretched) respectively. The configuration state (ρ, ϕ, ζ) for the pre-buckled hyperelastic vessel is well-established in literature, but the buckled state is not. To map the thin-walled vessel to the buckled state (ρ', ϕ', ζ') we perturb the stable pre-buckled state with a small displacement vector field independent of the radius.

The first solution method constructs the total potential energy functional mapping the set of all continuous bounded functions on $\mathbb{R}^3$ to the set of all reals $\mathbb{R}$. The functions we seek are by definition the perturbations that transform the vessel from a pre-buckling state to a buckled state. This pre-buckling state is assumed to be near buckling at either a stable equilibrium point or at the neutral equilibrium. Taking the first variation, we seek functions (u, v, w) which extremize the potential energy functional Π. Those which maximize Π push the vessel into an unstable state, whereas those which minimize Π either maintain stable equilibrium or maintain neutral equilibrium. If the critical point is a stationary inflection point then the vessel is pushed from a stable equilibrium to a neutral equilibrium. The specific critical point can be confirmed by the second variation and in general we seek $\delta^2\Pi < 0$.

The next solution methods both involve an approximation of $\delta\Pi$ using a Taylor series. The pre-buckling state in this case is taken to be the moment at which buckling occurs, i.e. the bifurcation point, and the buckled state is the moment just after buckling. By taking the difference between these two states we obtain $\Delta\Pi$ which, since the states are so close together, is a good approximation of the first variation of Π. We will seek the state $\Delta\Pi = 0$ so that we approximate the extremum of the functional. Note the displacement field will therefore necessarily cause a zero net change in energy since the functional is already maximized, or may cause a negative change in energy as the vessel becomes unstable. Hence it is enough that our displacement functions simply obey equilibrium, i.e. $\delta\Pi = 0$.

After deriving $\Delta\Pi$ we construct what we call the modified equations of equilibrium from elasticity theory. After we prove some necessary properties about the potential energy functional we outline a method for a solution similar to that used in our previous work. For reference the elasticity solutions to the static equilibrium with traction boundary conditions are outlined here in this work. The third solution method utilizes the kernel of a linear transformation which characterizes the mapping of the buckling displacement field amplitudes when equilibrium is attained and the boundary conditions are satisfied. The linear transformation is taken from our previous work and is the matrix $\mathbf{M}^t$ which includes traction boundary conditions. The kernel represents all possible displacement amplitudes which satisfy static equilibrium and the boundary condition, so we can substitute these into the $\Delta\Pi$ equation to obtain an relation for buckling eigenmodes, material parameters, and geometric parameters.

Chapter 2: REVIEW

2.1 State of Current Research

The motivation for this model is some previous work by Han et al. who used elastic beam theory to characterize the buckling of arteries subjected to large internal pressures and axial tension [11] [12]. Such a simplification in applicable theory was allowed due to the loading symmetry thereby projecting the three-dimensional problem into one dimension. Imposing a moment eliminates this type of symmetry during buckling and therefore an equivalent model for a triaxially loaded vessel must account for all geometric nuances by having a buckling displacement vector field dependent on multiple dimensions.

The energy method used by Han was not viable as a starting point since it required previous work to justify the usage of a linear elastic bending stiffness for the vessel [12]. However, experiments with great saphenous veins did indicate an approximately linear torsion-angle of twist loading curve so a future project could undertake this approach and use linear elastic torsion theory as an approximation [10]. Although Han's earlier work used linear elastic beam theory and small strain approximations, his later work used finite deformation theory with the Fung constitutive model applied to the linear beam solution method used in his previous model [13]. The second model relied on a first-order approximation of buckling deformation terms which is the same approximation used in this work and may not be valid for large deformations.

Preconditioned arteries have been successfully fit to the anisotropic Fung constitutive model for several different animals [4] [9] [11]. The original intention of this work was to derive a torsional buckling model for arteries, however the thin-wall requirement became a necessary assumption in simplifying the resultant equations. We therefore chose to model veins which typically have a thinner wall but exhibit mechanical properties similar to arteries [10]. In hindsight it may have been better to choose a constitutive model which had more supporting publications for the vessels used and where fewer regressed parameters were required, such as the two-fiber model [10]. However, by tackling what is objectively a more convoluted and challenging model, we provide a framework which extends to more simplified models.

The torsional buckling of blood vessels was observed during *in vitro* testing conditions when subjecting arteries and veins to *in vivo* internal pressure and axial tension and artificial amounts of torsion [9] [10]. The clinical application is therefore primarily limited to niche grafting or excision surgical procedures which require blood vessels to be repositioned, although some patients have shown rare occurrences of vein kinking *in vivo* [28]. Although perhaps intuitive that blood vessels should not be twisted, it may be useful to lay the theoretical groundwork for a body of knowledge in which specific surgical limitations can be imposed based on vessel material properties and dimensions.

Beyond the clinical application, perhaps the more significant question is how far are we able to push classic linear elastic theory solution methods when examining nonlinear phenomena? Are first-order perturbations of a stable configuration sufficient to characterize and model mechanical failure? To what order must we evaluate perturbational displacements so that the subsequent partial differential equations describe enough of the vessels behavior? Would conditionally accurate quantitative results imply that the perturbation size is sufficiently small and that a linearization of the partial differential equations is sufficient to gain a reasonable approximation for certain types of vessels?

Torsional buckling models for linear elastic materials are quite complicated and sometimes require sophisticated solution techniques borrowed from applied mathematics [5] [8] [19] [24]. Full triaxial loading models are even more rare and are not attempted using elastic theory but instead shell theory [3]. Typically used are the shell theories proposed by Lloyd Hamilton Donnell and Wilhelm Flugge [5] [8]. They are sophisticated in derivation and have a host of applications to the mechanical failure of all types of loading conditions on various geometries such as plates, cylinders, and spheres. The solution methods are convoluted but simultaneously elegant in their approximations, and they are remarkably effective. However, the brilliance of shell theory is also its greatest drawback in that it is a steep learning curve understanding the intricacies of the force and moment balances along the shell wall. A more intuitive approach may be to consider mechanical failure to be a perturbation of a stable configuration described by traditional elasticity from which we can reformulate the subsequent equations of equilibrium and the total potential energy functional to apply for shells.

The general solution method seen in the works of Kardomateas at Georgia Tech University shows the elasticity equilibrium approach in detail but examines external pressure loading conditions on thick-wall orthotropic cylinders whose buckling patterns necessarily depend on the radius [19] [20]. Theoretical results are difficult for these models but in the case of combined axial compression and external pressure the solution method follows that from shell theory wherein a displacement field is prescribed [19]. More relevant to our work, Kim et al. (1999) used the approach outlined by Kardomateas to derive a torsional buckling model using the Galerkin method and displacement fields which utilized both Fourier and Legendre polynomial orthonormal bases to obtain theoretical and numerical solutions [21]. The application of a moment invalidates the symmetries used in the previous works by Kardomateas and complicates the resultant equilibrium equations to the point that other types of loading are excluded. A shell formulation under elasticity theory would be even more simple, more intuitive, and easier to solve and perhaps allow more sophisticated loading conditions. Furthermore, a shell formulation of the potential energy functional would allow some nonlinearity in the strain energy density function during buckling.

Innovation is driven by necessity, and since interest in buckling is becoming more and more scarce there is a lack of need for innovation. Most modern research on the buckling of materials involve carbon nanotubes for which linear elastic shell theory is more than adequate but whose complications are sourced in the application of molecular forces [25] [26] [27]. Additionally, thanks to massive leaps in computational processing power and software development, the use of finite-element simulations makes the development of sophisticated nonlinear buckling theories almost moot from a practical perspective. The buckling of materials due to its roots in bifurcation theory and partial differential equations is sophisticated by nature and therefore the effort required to produce a stable working theory is not worth what could otherwise be spent on simulating small, heavily targeted applications. In today's publish or perish academic atmosphere there is little motivation to develop these complicated theories, even if the intellectual merit of doing so speaks for itself.

The consequence is sparse present-day publications covering the nonlinear buckling of materials from a theoretical perspective. Simple loading of rubber in various geometries has seen some analysis simply due to the large applicability of rubber, and there are some publications on the nonlinear buckling of neo-Hookean columns and beams [1] [2] [6]. Ertepinar et al. (1975) used a solution method for the torsional buckling of a hyperelastic rubber-like material similar to the one used in our previous work, however it makes use of stress functions whereas none are presented here [6]. Another example of a type of nonlinear buckling analysis is in graded linear elastic materials whether by temperature or mechanical properties [16]. In this case variational calculus is often used in the context of an energy conservation equation relating the work to the potential energy of the system.

In this light there is still a need for a nonlinear buckling theory valid for thick-walled vessels, even if the application is narrow in scope. There are nonlinear shell theories but scarce publications utilizing them likely due to their complexity [15] [23]. It is with this motivation in mind by which we undergo the extensive and tedious derivations in this work although the application is strictly to thin-walled vessels. Deriving a framework from first principles which implements challenging loading configurations, even if the theory requires somewhat of a linearization process, allows a top-down understanding of simpler cases without the use of symmetries and without the use of convoluted linear elastic shell theories which may not apply to hyperleastic materials, or nonlinear shell theories which may not have analytical solutions.

It is the hope of this and our other work to encourage further research on nonlinear mechanical failure theories. Better understanding nonlinear phenomena is arguably humanity's next knowledge hurdle and it will likely define a lot of the mathematics of the 21st century. Our current solution techniques in solving nonlinear partial differential equations are very limited in the types of problems they can solve. Countless phenomena in nature are (precisely) modeled by nonlinear partial differential equations and our usage of nonlinear materials is only growing, so it is unfortunate that our ability to utilize these materials and model via simulations these phenomena is lagged so far behind by our ability to produce analytical solutions from first principles.

2.2 Continuum Mechanics Preliminaries

Consider a body (continuum) which undergoes a continuous deformation from a reference configuration to a deformed configuration in $\mathbb{R}^3$. Each particle P_i in the undeformed state can be represented by some vector $\vec{X}$ and is mapped to a point p_i in the deformed state with new vector $\vec{x}$. We define a displacement vector field in the Lagrangian description (i.e. with respect to the reference configuration) as

$$\vec{u}(\vec{X},t) = \vec{x}(\vec{X},t) - \vec{X} \tag{2.1}$$

where $\vec{u}(\vec{X},t)$ and $\vec{x}(\vec{X},t)$ are continuous and differentiable at all points within the undeformed continuum.

2.2.1 Jacobian Matrix and Determinant

Since $\vec{x}(\vec{X},t)$ is continuous and differentiable we can construct its Jacobian matrix $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$. The Jacobian matrix is a linear transformation and maps a vector $\vec{X}$ to the best pointwise linear approximation of $\vec{x}(\vec{X},t)$ at $\vec{X}$.

$$\mathbf{F} = \begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{bmatrix} \tag{2.2}$$

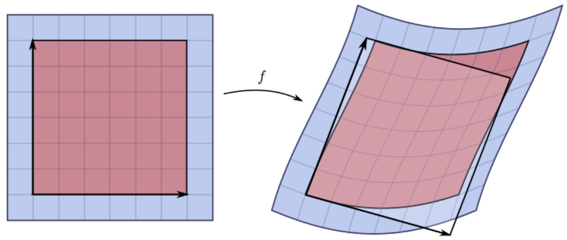

Figure 2.1: Jacobian linear approximation of a map $f : \mathbb{R}^2 \to \mathbb{R}^2$ [18]

Using the Jacobian matrix we map an undeformed differential line element element $d\vec{X}$ to a deformed $d\vec{x}$ in what can be see as essentially the total differential for vector functions

$$d\vec{x} = \frac{\partial \vec{x}}{\partial \vec{X}} d\vec{X} = \mathbf{F}(\vec{X}, t) d\vec{X} \tag{2.3}$$

The absolute value of the Jacobian determinant $|J|$ describes the volumetric dilatation of the mapping and the sign describes orientation ($J < 0$ reverses orientation). For incompressible materials $|J| := 1$ and in general in finite strain theory $J \neq 0$ at every vector $\vec{X}$ so the inverse function theorem guarantees the existence of a local inverse in a small neighborhood around each $\vec{X}$ and hence everywhere.

2.2.2 Green-Lagrange Strain

The strain in a material is a unitless measurement of deformation and characterizes the stretch or contraction a material experiences. In order to exclude the effect of rotation (the inverse of a rotation matrix is its transpose) we construct the right Cauchy-Green deformation tensor $\mathbf{C} = \mathbf{F}^T \mathbf{F}$ to be used in measurements of strain. The following strain tensor is known as the Green-Lagrange strain

$$\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I}) = \frac{1}{2}[(\nabla \vec{u})^T + \nabla \vec{u} + (\nabla \vec{u})^T \cdot \nabla \vec{u}] \tag{2.4}$$

In infinitesimal strain theory the quadratic terms $(\nabla \vec{u})^T \cdot \nabla \vec{u}$ are negligible. The Jacobian matrix $\mathbf{F}$ of the material volume mappings can be used to elegantly represent the large deformation Green-Lagrange strain $E_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i} + u_{m,i}u_{m,j})$ in matrix form as

$$\mathbf{E} = \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{I}) \tag{2.5}$$

2.2.3 Hyperelasticity and Stress Measures

A hyperelastic material is one in which the constitutive model is derived from a strain energy density function Ψ. Strain energy density functions are used to represent constitutive models for nonlinear materials instead of the traditional stress-strain relations seen in small deformation theory. The strain energy density function describes the strain energy density of the material at any given point and is composed of material parameters and strain components. It is from the strain energy density function which we derive the stresses within the material.

The Cauchy stress $\boldsymbol{\sigma}$ is the true stress of the deformed body and characterizes the forces applied on a differential volumetric element of a material. The stresses that develop within the body are those required to maintain each differential element's static equilibrium when balancing external forces. The Cauchy stresses are these counter-balancing forces normalized by the deformed area. For a hyperelastic material the Cauchy stress has the following form

$$\boldsymbol{\sigma} = \frac{1}{J}\mathbf{F} \cdot \frac{\partial \Psi}{\partial \mathbf{E}} \cdot \mathbf{F}^T \tag{2.6}$$

where J is the Jacobian determinant and corresponds to vessel diliatation, i.e. compressibility.

By constrast, the nominal stresses are the same counter-balancing internal forces normalized by the undeformed area (before deformation) and has the following form for a hyperelastic material

$$\mathbf{S} = \mathbf{F}^{-1}\boldsymbol{\sigma} = \frac{\partial \Psi}{\partial \mathbf{E}} \cdot \mathbf{F}^T \tag{2.7}$$

2.2.4 Fung Strain Energy Density Function

Some of the more advanced nonlinear constitutive models seem to derive meaning from primarily a numerical standpoint where several material parameters are optimized so that assumptions (such as boundary conditions) are strictly satisfied. It works well within a given range of the material deformation but leaves much to be desired if we are to try to interpret each individual parameter as part of the macroscopic vessel, even in the context of similarly regressed vessels.

The Fung strain energy density function Ψ with $\mathbf{E} = \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{I})$ is as follows

$$
\begin{aligned}
\Psi &= \frac{1}{2}C_0(e^Q - 1) \qquad (2.8)\\
Q &= b_1E_{RR}^2 + b_2E_{\Theta\Theta}^2 + b_3E_{ZZ}^2 + 2b_4E_{RR}E_{\Theta\Theta} + 2b_5E_{\Theta\Theta}E_{ZZ} + 2b_6E_{ZZ}E_{RR} + b_7(E_{R\Theta}^2 + E_{\Theta R}^2)\\
&\quad b_8(E_{\Theta Z}^2 + E_{Z\Theta}^2) + b_9(E_{RZ}^2 + E_{ZR}^2)
\end{aligned}
$$

In the case of a torsionally loaded, inflated, and stretched cylindrical vessel $E_{R\Theta} = E_{\Theta R} = E_{RZ} = E_{ZR} = 0$ thereby neutralizing the impact of parameters b_7 and b_9.

2.2.5 Equilibrium

If the material is in static equilibrium then we require two conditions to be met. One is the rotational static equilibrium

$$\Sigma M = 0 \qquad (2.9)$$

for any differential element such that there is no net angular acceleration. This result implies that the Cauchy stress is necessarily symmetric. The second condition is that

$$\nabla \cdot \boldsymbol{\sigma} = 0 \qquad (2.10)$$

so that there is no net acceleration of the differential element in any direction. It can be seen as an analog of the sums of the external forces $\Sigma\vec{F} = 0$ on a body at rest and can be derived by a simple force balance. The formulation of stress used in this paper is the nominal stress $\mathbf{S} = \frac{\partial\Psi}{\partial\mathbf{E}}\mathbf{F}^T$ which unlike the Cauchy stress tensor is not symmetric. It is used because the static equilibrium condition is the same as the Cauchy stress

$$\nabla \cdot \mathbf{S} = 0 \tag{2.11}$$

yet the calculations are much simpler.

2.2.6 Boundary Conditions and Nanson's Relations

The nominal stress has advantages in requiring fewer computational steps than the Cauchy stress, but there can be some disadvantages when applying boundary conditions. The traction conditions corresponding to the equilibrium equations above for the nominal stress tensor are as follows

$$\vec{t} = \mathbf{S} \cdot \hat{N} \tag{2.12}$$

where $\vec{t}$ is the applied external force vector normalized by its acting deformed area and $\hat{N}$ is the undeformed exterior surface normal vector. By contrast, the Cauchy stress traction conditions are

$$\vec{t} = \boldsymbol{\sigma} \cdot \hat{n} \tag{2.13}$$

where $\hat{n}$ is the deformed surface normal vector. Since we have undeformed and deformed normal surface vectors within one equation in the nominal stress boundary conditions, we require Nanson's relations

$$\hat{n} dA^{(1)} = J dA^{(0)} \hat{N} \cdot \mathbf{F}^{-T} \tag{2.14}$$

Nanson's relations map the surface normal $\hat{N}$ of an undeformed area $dA^{(0)}$ to the surface normal $\hat{n}$ of a deformed area $dA^{(1)}$. Curiously, we can solve for the dilation of every surface upon buckling by comparing the Cauchy stress boundary conditions to the nominal stress boundary conditions.

$$\mathbf{S} \cdot \hat{N} = \vec{t} = \boldsymbol{\sigma} \cdot \hat{n} = \boldsymbol{\sigma} \cdot \frac{dA^{(0)}}{dA^{(1)}} \hat{N} \cdot \mathbf{F}^{-T} \tag{2.15}$$

Computation of the Cauchy stresses is simple if we have already computed the nominal stresses, so it is not much more work to compare both sets of boundary conditions.

2.3 Variational Calculus Preliminaries

The calculus of variations is useful for solving nonlinear partial differential equations. It draws on techniques from nonlinear functional analysis and defines a class of problems which can be solved in this manner known as "variational problems." [7] Essentially we can consider a nonlinear partial differential equation to be the derivative of what is termed an energy functional. Therefore solutions to the differential equations are those functions which extremize the energy functional.

Let $U \subset \mathbb{R}^N$ be a bounded, open set with smooth boundary ∂U. We define the Lagrangian $L : \mathbb{R}^N \times \mathbb{R} \times \bar{U} \to \mathbb{R}$ in the context of a functional $I[w] : \mathbb{R}^N \to \mathbb{R}$ as

$$I[w] := \int_U L(\nabla w(x), w(x), x)\, dx \tag{2.16}$$

for $x \in U$, $w \in \mathbb{R}$, and $\nabla w(x) \in \mathbb{R}^N$. We assume that $w : \bar{U} \to \mathbb{R}$ are smooth functions satisfying a boundary condition $w = g$ on ∂U [7]. We define the first variation of the Lagrangian L as

$$\delta L := \alpha[\lim_{\alpha \to 0} \frac{dL(\nabla(w + \alpha v), w + \alpha v)}{d\alpha}] = \alpha[\lim_{\alpha \to 0} \frac{\Delta L}{\alpha}] \tag{2.17}$$

where v is an arbitrary function which satisfies the homogenous geometric boundary conditions $v = 0$ on ∂U. The first variation of the functional $I[w]$ is therefore

$$\delta I[w] = \delta \int_U L(\nabla w(x), w(x), x)\, dx = \int_U \delta L\, dx \tag{2.18}$$

After minimizing the expression above and integrating by parts, we can use the fundamental lemma of variational calculus to derive a partial differential equation from the integrand. The lemma is as follows

Lemma 1. *If f is a continuous, multivariable function on an open set $U \subset \mathbb{R}^N$ satisfying $\int_U f(x)h(x)\, dx = 0$ for all compactly supported smooth functions on U, then f is identically zero.*

The subsequent partial differential equation is known as the Euler-Lagrange equation. Consider $x \in U \subset \mathbb{R}^N$, U closed and bounded, and $w : \mathbb{R}^N \to \mathbb{R}^N$ with $w = (w^{(1)}, w^{(2)}, \cdots, w^{(n)})$. Then for $L = L(\nabla w^{(1)}, \cdots, \nabla w^{(n)}, w^{(1)}, \cdots, w^{(n)}, x)$ the subsequent Euler-Lagrange equations are as follows [22]

$$\begin{aligned}
0 =& \frac{\partial L}{\partial w^{(1)}} - \frac{\partial}{\partial x_1}(\frac{\partial L}{\partial w^{(1)}_{x_1}}) - \frac{\partial}{\partial x_2}(\frac{\partial L}{\partial w^{(1)}_{x_2}}) - \ \ldots\ - \frac{\partial}{\partial x_n}(\frac{\partial L}{\partial w^{(1)}_{x_n}}) \\
0 =& \frac{\partial L}{\partial w^{(2)}} - \frac{\partial}{\partial x_1}(\frac{\partial L}{\partial w^{(2)}_{x_1}}) - \frac{\partial}{\partial x_2}(\frac{\partial L}{\partial w^{(2)}_{x_2}}) - \ \ldots\ - \frac{\partial}{\partial x_n}(\frac{\partial L}{\partial w^{(2)}_{x_n}}) \\
&\vdots \\
0 =& \frac{\partial L}{\partial w^{(n)}} - \frac{\partial}{\partial x_1}(\frac{\partial L}{\partial w^{(n)}_{x_1}}) - \frac{\partial}{\partial x_2}(\frac{\partial L}{\partial w^{(n)}_{x_2}}) - \ \ldots\ - \frac{\partial}{\partial x_n}(\frac{\partial L}{\partial w^{(n)}_{x_n}})
\end{aligned} \tag{2.19}$$

Chapter 3: PRE-BUCKLING STATE

3.1 Assumptions

We will derive the Jacobian matrix for the deformation of a stress-free, thick-walled cylindrical vessel to a triaxially loaded state. The thick-walled vessel is assumed to be an incompressible straight cylinder of constant thickness that obeys the Fung hyperelastic constitutive model. We will assume that the ends are fixed so that any torsion and tension is applied uniformly at both ends, and that the pressure is applied uniformly within the vessel interior. We will also assume that the angle of twist deformation varies linearly across the vessel length in the unbuckled vessel and that the vessel thickness is unchanged during the application of torsion. Note that the loading is assumed to be axisymmetric but this is no longer applicable upon buckling. We will also assume that the stretch ratio of the vessel is held constant throughout torsional loading. However, this is an unrealistic assumption to make since the vessel necessarily contracts as it twists and therefore, although the vessel is held in place during *in vitro* testing, the vessel is constantly extending.

3.2 Configuration Mappings and Deformation Jacobian Matrix

The vector function mapping the stress-free state (R, Θ, Z) to the triaxially loaded state (ρ, ϕ, ζ) under our assumptions has the following components [17]

$$\rho = \rho(R, P, \cdots) \qquad \phi = \frac{\pi}{\Theta_0}\Theta + \frac{\gamma}{L}\Lambda Z \qquad \zeta = \lambda\Lambda Z \tag{3.1}$$

The vessel has undeformed length L and upon torsional loading an angle of twist γ. We are describing the full, non-truncated system so that the deformed radius and deformed wall-thickness are accurately modeled. The thin-wall condition will then be imposed on the thick-wall configuration when mapping from the pre-buckling state to the buckled state. The loaded vessel radius ρ is a function of pressure, the material parameters, vessel geometry, and previous radius location.

It is assumed to be independent of torsion so that the vessel radius does not shrink under torsional loading. Residual stresses are required to hold the vessel in the unloaded state and are noticeable when slicing the unloaded vessel. The term Θ_0 describes the opening angle due to residual stress as seen in Figure 1.3 and is taken to be π in the absence of residual stress. The stretch ratio $\lambda\Lambda$ is composed of a residual stress stretch Λ and another stretch λ to put the vessel under tension. Similarly we take Λ = 1 if neglecting the residual stress.

Per the definition of the Jacobian matrix, computing the partial derivatives of the configuration mappings results in the following deformation gradient **F** which maps the undeformed vessel to the fully deformed vessel

$$\mathbf{F} = \begin{bmatrix} \frac{\partial \rho}{\partial R} & 0 & 0 \\ 0 & \frac{\rho\pi}{R\Theta_0} & \rho\frac{\gamma}{L}\Lambda \\ 0 & 0 & \lambda\Lambda \end{bmatrix}$$

The thick-wall vessel is assumed incompressible thereby requiring that the Jacobian determinant be exactly one to restrict dilatation during mapping. Utilizing this condition we obtain a relation for $\frac{\partial \rho}{\partial R}$

$$\mathbf{F} = \begin{bmatrix} \frac{R\Theta_0}{\rho\pi\lambda\Lambda} & 0 & 0 \\ 0 & \frac{\rho\pi}{R\Theta_0} & \rho\frac{\gamma}{L}\Lambda \\ 0 & 0 & \lambda\Lambda \end{bmatrix}$$

3.3 Static and Hydrostatic Equiliibrium

If we are to use the Cauchy stress measure $\boldsymbol{\sigma} = \mathbf{F}\frac{\partial\Psi}{\partial\mathbf{E}}\mathbf{F}^T$ then the condition of static equilibrium with no acting body forces is given by $\nabla \cdot \boldsymbol{\sigma} = 0$. In the axially symmetric loading case in cylindrical coordinates we have the following coupled partial differential equations

$$\begin{aligned}
0 &= \frac{\partial\sigma_{rr}}{\partial R} + \frac{\partial H}{\partial R} + \frac{1}{R}(\sigma_{rr} - \sigma_{\theta\theta}) \\
0 &= \frac{\partial\sigma_{r\theta}}{\partial R} + \frac{1}{R}(\sigma_{r\theta} + \sigma_{\theta r}) \\
0 &= \frac{\partial\sigma_{rz}}{\partial R} + \frac{1}{R}\sigma_{rz}
\end{aligned} \tag{3.2}$$

where $H = H(R)$ is the hydrostatic pressure distribution within the vessel wall. The hydrostatic equilibrium of the vessel is taken from the radial equation of motion

$$\begin{aligned}
0 = R\frac{\partial H}{\partial R} &+ R\frac{\partial}{\partial R}[(\frac{R\Theta_0}{\rho\pi\lambda\Lambda})^2[\frac{b_1}{2}((\frac{R\Theta_0}{\rho\pi\lambda\Lambda})^2 - 1) + \frac{b_6}{2}(\frac{\rho^2\gamma^2}{L^2} + \lambda^2\Lambda^2 - 1) + \frac{b_5}{2}((\frac{\rho\pi}{R\Theta_0})^2 - 1)]C_0e^Q] \\
&+ (\frac{R\Theta_0}{\rho\pi\lambda\Lambda})^2[\frac{b_2}{2}((\frac{R\Theta_0}{\rho\pi\lambda\Lambda})^2 - 1) + \frac{b_6}{2}(\frac{\rho^2\gamma^2}{L^2} + \lambda^2\Lambda^2 - 1) + \frac{b_4}{2}((\frac{\rho\pi}{R\Theta_0})^2 - 1)]C_0e^Q \\
&- (\frac{\rho\pi}{R\Theta_0})^2[\frac{b_2}{2}((\frac{\rho\pi}{R\Theta_0})^2 - 1) + \frac{b_5}{2}(\frac{\rho^2\gamma^2}{L^2} + \lambda^2\Lambda^2 - 1) + \frac{b_4}{2}((\frac{R\Theta_0}{\rho\pi\lambda\Lambda})^2 - 1)]C_0e^Q \\
&- \frac{\rho^2\gamma^2}{L^2}[(b_3(\frac{\rho^2\gamma^2}{L^2} + \lambda^2\Lambda^2 - 1) + \frac{b_5}{2}((\frac{\rho\pi}{R_0\Theta_0})^2 - 1) + \frac{b_6}{2}((\frac{R_0\Theta_0}{\rho\pi\lambda\Lambda})^2 - 1)]C_0e^Q \\
&- 4b_8(\frac{\rho^2\pi\gamma}{R\Theta_0 L})^2C_0e^Q
\end{aligned} \tag{3.3}$$

It is from this relation by which we define an expression for the hydrostatic distribution. We will assume in this state that the pressure from the inside of the wall is perfectly dissipated through the arterial wall such that $\sigma_{rr}(R_o) + H(R_o) - \sigma_{rr}(R_i) - H(R_i) = 0 - (-p) = p$. An explicit analytical result for H is quite difficult for the thick-wall case. It should be noted that different types of loading will change the radial equation of motion, hence also changing the hydrostatic pressure distribution. As will be shown the hydrostatic pressure distribution upon buckling changes from that seen here due to the thin-wall assumption and due to the new stresses that arise during buckling.

3.4 Minimization of Objective Function

We obtain the material parameters for a Fung cylindrical vessel by relating the internal stresses to the external applied loads and defining an error objective function which must be minimized over all Fung material parameters. The following three equations are used in a torsionally loaded vessel

$$
\begin{aligned}
P_t &= \int_{\rho_i}^{\rho_o} \frac{1}{R}(\sigma_{\theta\theta} - \sigma_{rr})\, dR \\
N_t &= \pi \int_{\rho_i}^{\rho_o} 2\sigma_{zz} - \sigma_{rr} - \sigma_{\theta\theta}\, dR + \pi\rho_i^2 P_t \\
T_t &= 2\pi \int_{\rho_i}^{\rho_o} \sigma_{\theta z} R^2\, dR
\end{aligned} \tag{3.4}
$$

And the objective function G is given by

$$
G = (\frac{P_t - P_{exp}}{P_{exp}})^2 + (\frac{N_t - N_{exp}}{N_{exp}})^2 + (\frac{T_t - T_{exp}}{T_{exp}})^2 \tag{3.5}
$$

relating the theoretical and experimental loads.

3.5 Pre-buckling Radius ρ

We will treat the inflated, stretched, and twisted radius ρ as a parameter obtained from experimentation in the rest of this work, but an analytical relation for the radius in terms of loading conditions and material parameters can be approximated for the thin-wall case. We can obtain a general picture of ρ by restricting dilatation of the torsionally loaded but not critically loaded thick-wall vessel such that $det(F) := 1$

$$
\frac{\partial \rho}{\partial R} = \frac{\partial \rho}{\partial r}\frac{\partial r}{\partial R} = \frac{r}{\rho\lambda}\frac{R\Theta_0}{r\pi\Lambda} \tag{3.6}
$$

Upon integrating

$$
\rho = R\sqrt{\frac{\Theta_0}{\pi\lambda\Lambda} + \frac{c_1}{R^2}} \tag{3.7}
$$

After some algebra we have a relation that must be satisfied by matching positions in the loaded radius and stress-free radius of the vessel. For the interior and exterior of the undeformed and deformed vessel walls we have

$$c_1 = \rho_o^2 - \frac{\Theta_0}{\pi\lambda\Lambda}R_o^2 = \rho_i^2 - \frac{\Theta_0}{\pi\lambda\Lambda}R_i^2 \tag{3.8}$$

Using the above relation combined with one experimental measurement of either the interior or exterior radius during pressure inflation we can model the change in thickness of the wall assuming the unloaded vessel dimensions are known. In other words given R_o, R_i, and ρ_o we find ρ_i by

$$\rho_i = \sqrt{\rho_o^2 - \frac{\Theta_0}{\pi\Lambda\Lambda}(R_o^2 - R_i^2)} \tag{3.9}$$

There are a few ways to derive an analytical relation for ρ in terms of material parameters depending on what is known. Perhaps the easiest is to make the assumption that the vessel radius is the same before and after a torsional load is applied. With this assumption we can consider the loading state before torsion is applied and use the pressure equation relating the transmural pressure to the internal stresses.

Consider a thin-walled cylindrical vessel with radius $\rho_0 = \rho_0(\rho_i, \rho_o)$ which has been adapted from a thick-walled vessel with inner radius $\rho_i - \epsilon$ and outer radius $\rho_o + \epsilon$. There is a subtlety here that requires mention. If we are to take the midpoint of the undeformed vessel wall as the representative vessel location, it does not translate to the midpoint of the deformed vessel well since the thickness scales quadratically. We must therefore use relation (3.7) to obtain the proper location of ρ_0. If we take $R_0 = \frac{1}{2}(R_o + R_i)$ then we require that

$$\begin{aligned}\rho_0 &= \sqrt{\rho_o^2 - \frac{\Theta_0}{\pi\Lambda\Lambda}(R_o^2 - R_0^2)} \\ &= \sqrt{\rho_o^2 - \frac{\Theta_0}{\pi\Lambda\Lambda}[R_o^2 - \frac{1}{4}(R_o^2 + R_i^2 - 2R_iR_o)]}\end{aligned} \tag{3.10}$$

where R_i, R_o are the undeformed vessel inner and exterior radiuses respectively and ρ_o is the deformed vessel exterior radius.

We will consider a partially distributed pressure through the vessel wall (see Chapter 7 Sections 7.2 and 7.4) so that $(1-\nu)$ is the fraction of the pressure distributed within the wall thickness with $0 \leq \nu \leq 1$. If a nonzero remainder $\nu(p_{ext} - p_{int})$ is chosen then it is assumed to be distributed longitudinally since the vessel is thin-walled. Then

$$(1-\nu)(p_{ext} - p_{int}) = \int_{\rho_i}^{\rho_o} (\sigma_{\theta\theta} - \sigma_{rr}) \frac{1}{\rho} \, d\rho \implies \rho_o = \rho_i e^{\frac{(1-\nu)(p_{ext} - p_{int})}{\sigma_{\theta\theta} - \sigma_{rr}}} \tag{3.11}$$

where σ_{rr} and $\sigma_{\theta\theta}$ are functions of ρ_0. We can use equation (3.9) to solve for ρ_o in terms of ρ_0, R_i and R_o

$$\rho_o^2 = \frac{-\frac{\Theta_0}{\pi \Lambda \Lambda}(R_o^2 - R_i^2) e^{\frac{2(1-\nu)(p_{ext} - p_{int})}{\sigma_{\theta\theta} - \sigma_{rr}}}}{(1 - e^{\frac{2(1-\nu)(p_{ext} - p_{int})}{\sigma_{\theta\theta} - \sigma_{rr}}})} \tag{3.12}$$

If we wish to solve for ρ_o in terms of only R_0 we require the axial load equation to substitute for the interior deformed radius in equation (3.10) . The axial tension equation gives us

$$N - \pi \rho_i^2 p_{int} = \pi \int_{\rho_i}^{\rho_o} (2\sigma_{zz} - \sigma_{rr} - \sigma_{\theta\theta}) \rho \, d\rho \implies \rho_i = [\rho_o^2 - \frac{2(N - \pi \rho_i^2 p_{int})}{\pi(2\sigma_{zz} - \sigma_{rr} - \sigma_{\theta\theta})}]^{\frac{1}{2}} \tag{3.13}$$

where σ_{zz} is a function of ρ_0 and we of course took the positive root. Therefore

$$\rho_o = \frac{2(N - \pi \rho_i^2 p_{int})}{\pi(2\sigma_{zz} - \sigma_{rr} - \sigma_{\theta\theta})(1 - e^{\frac{2(1-\nu)(p_{ext} - p_{int})}{\sigma_{\theta\theta} - \sigma_{rr}}})} \tag{3.14}$$

If the unloaded vessel thickness was measured and if we assume the vessel thickness is unchanged upon loading then in theory we can obtain the inner radius ρ_i and hence ρ_0. Analytically solving for ρ_0 will require a lot of tedious algebra since the stresses are themselves functions of $\rho_0(\rho_i, \rho_o)$. In this work we define ρ_0 to be the deformed vessel midpoint.

Alternatively to above we can use the torsion equation if we know the angle of twist of the vessel

$$T = 2\pi \int_{\rho_i}^{\rho_o} \sigma_{\theta z} \rho^2 d\rho \implies \rho_o = (\rho_i + \frac{3T}{2\pi\sigma_{\theta z}})^{\frac{1}{3}} \tag{3.15}$$

The above could be useful if we are to remove the assumption that vessel thickness does not change upon twisting but removing said assumption would require different configuration mappings.

We can illustrate further what these equations will look like by computing the stresses explicitly. Under these loading conditions and using ρ_0 the configuration mappings will be

$$\rho_0 = \rho_0(P, N, \cdots) \qquad \phi = \frac{\pi}{\Theta_0}\Theta + \frac{\gamma}{L}\Lambda Z \qquad \zeta = \lambda\Lambda Z \tag{3.16}$$

and the subsequent strain components

$$\begin{aligned} 2E^p_{rr} &= (\frac{R_0\Theta_0}{\rho_0\pi\lambda\Lambda})^2 - 1 & 2E^p_{\theta z} &= \frac{\rho_0^2\pi\gamma}{R_0\Theta_0 L} \\ 2E^p_{\theta\theta} &= (\frac{\rho_0\pi}{R_0\Theta_0})^2 - 1 & 2E^p_{zz} &= \frac{\rho_0^2\gamma^2\Lambda^2}{L^2} + \lambda^2\Lambda^2 - 1 \end{aligned} \tag{3.17}$$

The subsequent pre-buckling Cauchy stresses are given in Appendix A.1. If we consider the pre-torsion configuration state we can reduce the stress terms by setting $\gamma := 0$. We can abstract the form of the stresses by considering constants k_{ij} and polynomials $s_{ij}(\rho_0)$ and $q(\rho_0)$.

$$\begin{aligned} \sigma_{rr} &= [s_{rr}(\rho_0^{-4}, \rho_0^{-2}) + k_{rr}]e^{q(\rho_0^{-4}, \rho_0^{-2}, \rho_0^2, \rho_0^4) + k_q} \\ \sigma_{\theta\theta} &= [s_{\theta\theta}(\rho_0^{-2}, \rho_0^2, \rho_0^4) + k_{\theta\theta}]e^{q(\rho_0^{-4}, \rho_0^{-2}, \rho_0^2, \rho_0^4) + k_q} \\ \sigma_{zz} &= [s_{zz}(\rho_0^{-2}, \rho_0, \rho_0^2, \rho_0^3) + k_{zz}]e^{q(\rho_0^{-4}, \rho_0^{-2}, \rho_0^2, \rho_0^4) + k_q} \\ \sigma_{\theta z} &= [s_{\theta z}(\rho_0^{-2}, \rho_0^2) + k_{\theta z}]e^{q(\rho_0^{-4}, \rho_0^{-2}, \rho_0^2, \rho_0^4) + k_q} \end{aligned} \tag{3.18}$$

Now taking the difference

$$\sigma_{\theta\theta} - \sigma_{rr} = [s'(\rho_0^{-4}, \rho_0^{-2}, \rho_0^2, \rho_0^4) + k_{\theta\theta} - k_{rr}]e^{q(\rho_0^{-4}, \rho_0^{-2}, \rho_0^2, \rho_0^4) + k_q}$$
$$2\sigma_{zz} - \sigma_{rr} - \sigma_{\theta\theta} = [s''(\rho_0^{-4}, \rho_0^{-2}, \rho_0, \rho_0^2, \rho_0^3, \rho_0^4) + 2k_{zz} - k_{rr} - k_{\theta\theta}]e^{q(\rho_0^{-4}, \rho_0^{-2}, \rho_0^2, \rho_0^4) + k_q} \quad (3.19)$$

Rearranging (3.14) and substituting we have

$$\frac{2(N - \pi\rho_i^2 p_{int})}{\pi\rho_o} = (s'' + 2k_{zz} - k_{rr} - k_{\theta\theta})(e^{q+k_q} - e^{q+k_q+2(1-\nu)(p_{ext}-p_{int})(s'+k_{\theta\theta}-k_{rr})^{-1}e^{-q-k_q}}) \quad (3.20)$$

Substituting $\rho_0(\rho_o)$ gives us polynomials over ρ_o and a very tedious resultant equation. However, this relation provides a rough analytical model for the radial displacement during inflation of a thin-walled vein as a function of its material parameters and the applied loads. The parameters ϵ and ν can be adjusted to accurately predict the expansion of the wall's midpoint ρ_0 using experimental data. This idea could be interesting to prove rigorously. The function ρ is typically considered a black-box for thick-wall deformation, but we can approximate the thin-wall vessel dilation analytically. This result is useful because if the stiffness coefficients of the vessel can be estimated or are already known then the rest of the input parameters required by the model are taken from an undeformed configuration.

Chapter 4: BUCKLED STATE

4.1 Assumptions

The thin-wall condition implies the stress distribution is uniform across the vessel thickness such that we can use some $R = R_0$, $\rho = \rho_0$ to replace R, ρ respectively so that integration over the radius becomes trivial. The rest of the assumptions for the pre-buckling state hold for the buckled state.

4.2 Configuration Mappings and Deformation Jacobian Matrix

We assume the following configuration mappings for the buckled state. It is a slightly modified thin-wall version of the thick-wall torsion configuration state outlined by Jay Humphrey in Cardiovascular Solid Mechanics (2002) with the addition of perturbation functions Ω, Φ, and η which form a displacement field that maps the vein from a stable state to a buckled state [17].

$$\rho' = \Omega(\Theta, Z) + \rho \qquad \phi' = \Phi(\Theta, Z) + \frac{\pi}{\Theta_0}\Theta + \frac{\gamma_0}{L_0}Z \qquad \zeta' = \eta(\Theta, Z) + \lambda\Lambda Z \tag{4.1}$$

For the analytical system to represent the experimental system we must slightly alter the angle of twist intensity. The intensity in the stable torsionally loaded state is given in terms of the undeformed length L but in the buckled case we wish to use the stretched and kinked segment L_0. If γ is the experimental angle of twist and intensity $\frac{\gamma}{L}$ then we define the global deformation angle of twist $\gamma_0 := \frac{(\alpha-\beta)\gamma}{\Lambda}$ so that $\frac{\gamma_0}{L_0} = \frac{(\alpha-\beta)\gamma}{(\alpha-\beta)\lambda\Lambda^2 L} = \frac{\gamma}{\lambda\Lambda^2 L}$. This implies that at vessel length L_0 the angle of twist is the same as the angle of twist of the pre-buckled system evaluated at αL but translated by the angle of twist at βL to zero the function ϕ'. We have added an extra Λ term in the denominator so that we do not have to continue rewriting the term when it will be taken to be one.

The Jacobian matrix of the deformation from state (R, Θ, Z) to (ρ', ϕ', ζ') can be derived by assuming a thick-wall deformation from (R, Θ, Z) to (ρ, ϕ, ζ) and then thin-wall from (ρ, ϕ, ζ) to

(ρ', ϕ', ζ') such that E_{RR} is nonzero

$$\mathbf{F} = \begin{bmatrix} \frac{R_0\Theta_0}{\rho_0\pi\lambda\Lambda} & \frac{1}{R_0}\frac{\partial\Omega}{\partial\Theta} & \frac{\partial\Omega}{\partial Z} \\ 0 & \frac{\rho_0+\Omega}{R_0}(\frac{\partial\Phi}{\partial\Theta}+\frac{\pi}{\Theta_0}) & (\rho_0+\Omega)[\frac{\partial\Phi}{\partial Z}+\frac{\gamma_0}{L_0}] \\ 0 & \frac{1}{R_0}\frac{\partial\eta}{\partial\Theta} & \frac{\partial\eta}{\partial Z}+\lambda\Lambda \end{bmatrix}$$

If we exclude a Lagrangian multiplier as representation of a hydrostatic pressure distribution within the buckled vessel there will be problems in the static equilibrium equations. Namely, there is a hidden conservation equation in the first equilibrium equation that must be satisfied

$$\begin{aligned} 0 = [b_1(\frac{R_0^2\Theta_0^2}{\rho_0^2\pi^2\lambda^2\Lambda^2}-1)+b_4(\frac{\rho_0^2\pi^2}{R_0^2\Theta_0^2}-1)+b_6(\frac{\rho_0^2\gamma_0^2}{L_0^2}+\lambda^2\Lambda^2-1)]\frac{R_0\Theta_0}{\rho_0\pi\lambda\Lambda}-[b_4(\frac{R_0^2\Theta_0^2}{\rho_0^2\pi^2\lambda^2\Lambda^2}-1) \\ +b_2(\frac{\rho_0^2\pi^2}{R_0^2\Theta_0^2}-1)+b_5(\frac{\rho_0^2\gamma_0^2}{L_0^2}+\lambda^2\Lambda^2-1)\frac{\rho_0\pi}{R_0\Theta_0}]-b_8(\frac{\rho_0^2\pi\gamma_0}{R_0\Theta_0L_0})\frac{\rho_0\gamma_0}{L_0} \end{aligned} \quad (4.2)$$

This equation invalidates the assumption that ρ_0 does not depend on γ_0 and if computed with experimental values is a poor approximation, so this approach is not valid. It may be tempting instead to consider the vessel to be thin-wall from the reference state to the buckled state such that

$$\mathbf{F} = \begin{bmatrix} 1 & \frac{1}{R_0}\frac{\partial\Omega}{\partial\Theta} & \frac{\partial\Omega}{\partial Z} \\ 0 & \frac{\rho_0+\Omega}{R_0}(\frac{\partial\Phi}{\partial\Theta}+\frac{\pi}{\Theta_0}) & (\rho_0+\Omega)[\frac{\partial\Phi}{\partial Z}+\frac{\gamma_0}{L_0}] \\ 0 & \frac{1}{R_0}\frac{\partial\eta}{\partial\Theta} & \frac{\partial\eta}{\partial Z}+\lambda\Lambda \end{bmatrix}$$

However this mapping also causes its share of problems. Although several strain and stress terms cancel out, there is still a constant from the $S_{\Theta\Theta}$ term that must be zero in the radial equation of motion. This is an even harsher restriction than in the previous case. Both methods necessarily require a Lagrangian multiplier, so we will extensively treat its tedious existence when we examine the buckled partial differential equations of equilibrium.

As for which pre-buckling treatment to use (thin- or thick-wall) we wish to accurately model the vessel dilation and thickness change during pressure inflation so we will use a thick-wall deformation up until the moment the vessel buckles at which point the vessel experiences a thin-wall deformation. We will not make any assumptions about the volume of the material or its compressibility during buckling.

In order to zero the constant terms in the equilibrium equations we will sacrifice the pure thin-wall assumption during buckling so that we can apply the Lagrangian multiplier H. However, we will show the static equilibrium equations both with and without a pressure distribution within the wall. In the former case we will prove that the pressure distribution within the wall is quite small and dependent only on the radius. There is a subtlety in the amount of pressure distributed within the wall such that if we are to treat the wall as having a thickness, there is necessarily a maximum thickness corresponding to that distance which completely dissipates the pressure applied to the vessel interior.

4.3 Strain and Strain Energy

The Fung hyperelastic strain energy density function, with $\mathbf{E} = \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{I})$, is as follows

$$
\begin{aligned}
\Psi &= \frac{1}{2}C_0(e^Q - 1) \\
Q &= b_1 E_{RR}^2 + b_2 E_{\Theta\Theta}^2 + b_3 E_{ZZ}^2 + 2b_4 E_{RR}E_{\Theta\Theta} + 2b_5 E_{\Theta\Theta}E_{ZZ} + 2b_6 E_{ZZ}E_{RR} + b_7(E_{R\Theta}^2 + E_{\Theta R}^2) \\
&\quad b_8(E_{\Theta Z}^2 + E_{Z\Theta}^2) + b_9(E_{RZ}^2 + E_{ZR}^2)
\end{aligned}
\tag{4.3}
$$

We will add the Lagrangian multiplier term H onto each normal stress term retroactively instead of including it into the strain energy density function Ψ as there is no elegant solution like in the biaxially loaded case. Fung and Chuong (1983) came up with a formulation for Ψ that uses incompressibility to add H onto normal stresses automatically upon taking the derivative, but the inclusion of shear stresses in our problem makes this difficult without grabbing a function out of thin air that arbitrarily satisfies the properties we need [4].

As outlined in the introduction, each component of the displacement field is considered to have sufficiently small amplitude such that any displacement terms of order two or greater can be considered negligible. After reducing small terms the strain components reduce to

$$\begin{aligned}
2E^b_{RR} &= (\frac{R_0\Theta_0}{\rho_0\pi\lambda\Lambda})^2 - 1 \\
2E^b_{R\Theta} &= \frac{\Theta_0}{\rho_0\pi\lambda\Lambda}\frac{\partial\Omega}{\partial\Theta} \\
2E^b_{RZ} &= \frac{R_0\Theta_0}{\rho_0\pi\lambda\Lambda}\frac{\partial\Omega}{\partial Z} \\
2E^b_{\Theta\Theta} &= \frac{\rho_0^2\pi}{R_0^2\Theta_0}(2\frac{\partial\Phi}{\partial\Theta} + \frac{\pi}{\Theta_0}) + 2\frac{\rho_0\pi^2}{R_0^2\Theta_0^2}\Omega - 1 \\
2E^b_{\Theta Z} &= \frac{\rho_0^2}{R_0}(\frac{\gamma_0}{L_0}\frac{\partial\Phi}{\partial\Theta} + \frac{\pi}{\Theta_0}\frac{\partial\Phi}{\partial Z} + \frac{\pi\gamma_0}{\Theta_0 L_0}) + \frac{2\rho_0\pi\gamma_0}{R_0\Theta_0 L_0}\Omega + \frac{\lambda\Lambda}{R_0}\frac{\partial\eta}{\partial\Theta} \\
2E^b_{ZZ} &= (\rho_0^2 + 2\Omega\rho_0)\frac{\gamma_0^2}{L_0^2} + 2\frac{\rho_0^2\gamma_0}{L_0}\frac{\partial\Phi}{\partial Z} + 2\lambda\Lambda\frac{\partial\eta}{\partial Z} + \lambda^2\Lambda^2 - 1
\end{aligned} \tag{4.4}$$

After eliminating higher order terms for Q_b and Q_{pb} (buckled and prebuckling, respectively) taken from the Fung hyperelastic model, we can express the exponential component of the strain energy density function in terms of constants C_i and partial derivatives of the buckling displacement functions.

$$\begin{aligned}
e^{Q_b} &= e^{C_1}e^{C_2\Omega}e^{C_3\frac{\partial\Phi}{\partial\Theta}}e^{C_4\frac{\partial\Phi}{\partial Z}}e^{C_5\frac{\partial\eta}{\partial\Theta}}e^{C_6\frac{\partial\eta}{\partial Z}} \\
e^{Q_{pb}} &= e^{C_7}
\end{aligned} \tag{4.5}$$

Using the power series representations of the exponentials we can simplify the exponential nicely

$$\begin{aligned}
e^{Q_b} &= e^{C_1}\sum_{n_1=0}^{\infty}\frac{(C_2\Omega)^{n_1}}{n_1!}\sum_{n_2=0}^{\infty}\frac{(C_3\frac{\partial\Phi}{\partial\Theta})^{n_2}}{n_2!}\sum_{n_3=0}^{\infty}\frac{(C_4\frac{\partial\Phi}{\partial Z})^{n_3}}{n_3!}\sum_{n_4=0}^{\infty}\frac{(C_5\frac{\partial\eta}{\partial\Theta})^{n_4}}{n_4!}\sum_{n_5=0}^{\infty}\frac{(C_6\frac{\partial\eta}{\partial Z})^{n_5}}{n_5!} \\
&= e^{C_1}(1 + C_2\Omega + C_3\frac{\partial\Phi}{\partial\Theta} + C_4\frac{\partial\Phi}{\partial Z} + C_5\frac{\partial\eta}{\partial\Theta} + C_6\frac{\partial\eta}{\partial Z})
\end{aligned} \tag{4.6}$$

Hence

$$\Psi_b = \frac{1}{2}C_0[e^{C_1}(1 + C_2\Omega + C_3\frac{\partial\Phi}{\partial\Theta} + C_4\frac{\partial\Phi}{\partial Z} + C_5\frac{\partial\eta}{\partial\Theta} + C_6\frac{\partial\eta}{\partial Z}) - 1]$$

$$\Psi_{pb} = \frac{1}{2}C_0(e^{C_7} - 1) \tag{4.7}$$

Chapter 5: TOTAL POTENTIAL ENERGY FUNCTIONAL

5.1 Energy Conservation from Continuum Mechanics

The total potential energy functional has roots in the energy conservation of a continuum. A common formulation of energy conservation in continuum mechanics is the following

$$\rho\frac{\partial e}{\partial t} - \sigma : (\nabla v) + \nabla \cdot q - \rho s = 0 \tag{5.1}$$

where $\rho(x,t)$ is the mass density, $e(x,t)$ is the internal energy per unit mass, $\sigma(x,t)$ is the Cauchy stress, $v(x,t)$ is the particle velocity, q is the heat flux vector and s is the rate at which energy is generated by sources inside the volume (per unit mass). If we consider a material volume V and suppose that ρ is constant and there is no energy source or heat we can use another formulation

$$\int_A \vec{t}\cdot\vec{v}\,dA + \int_V \rho\vec{b}\cdot\vec{v}\,dV = \int_V \sigma \otimes \frac{\partial F}{\partial t}\,dV + \frac{\partial}{\partial t}\int_V \frac{1}{2}\rho\vec{v}\cdot\vec{v}\,dV \tag{5.2}$$

where $\vec{t}$ is the external traction vector. Here we again have a time-dependent equation. The two terms on the LHS represent the external work due to traction and the work due to body forces from left to right respectively. On the RHS from left to right we have the internal strain energy due to internal stresses and internal kinetic energy respectively. In our model we will neglect body forces and we will assume that the system is static such that there is no kinetic energy. Hence

$$\int_A \vec{t}\cdot\vec{v}\,dA = \int_V \sigma \otimes \frac{\partial E}{\partial t}\,dV \tag{5.3}$$

The RHS is the total strain energy of the vessel and seems complicated, but we conveniently have a strain energy density function w for our hyperelastic material.

$$\int_A \vec{t}\cdot\vec{v}\,dA = \int_V \frac{\partial w}{\partial t}\,dV \tag{5.4}$$

To wit, our system is independent of time. Since the above equation must hold for any time interval we have

$$\int_A \vec{t} \cdot \vec{u} \, dA = \int_V w \, dV \tag{5.5}$$

This relates the external work to the internal strain energy of the system and is the foundation of the energy functional that we will define and prove when extremized yields the body in equilibrium.

5.2 Total Potential Energy Functional

We define an energy functional $\Pi := U_s - W_{ext}$ to represent the total potential energy of the vessel at any configuration. The system is assumed to be stationary and without heat transfer so the only sources of energy are the external work and the internal potential energy of the material. In the case of a generalized system with strain energy density function Ψ and without body forces we have the following

$$\Pi = \int_V \Psi dV - \int_{S_t} \vec{t^n} \cdot \vec{u} \, dS \tag{5.6}$$

where $\vec{t^n}$ is the surface traction on surface S_t with normal $\hat{n}$ and $\vec{u}$ is the associated displacement caused by the traction.

Theorem 5.2.1. *If the first variation ofthe total potential energy functional of a compressible hyper-elastic material is minimized then the equations of static equilibrium* $\nabla \cdot S = 0$ *are automatically satisfied.*

Proof. The proof for linear elastic materials is well established but here we will show it also holds for a hyperelastic material without assumption of compressibility. We require a few relations derived from hyperelastic and large-deformation theory

$$\mathbf{S} = \frac{\partial \Psi}{\partial \mathbf{E}}\mathbf{F}^T \implies S_{ij} = \frac{\partial w}{\partial E_{Ki}} F_{jK} \qquad \boldsymbol{\sigma} = \mathbf{F}\mathbf{S} \implies \sigma_{ij} = F_{iK}\frac{\partial w}{\partial E_{KL}} F_{jL}$$
$$F_{ij} = \delta_{ij} + u_{i,j} \qquad E_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i} + u_{M,i}u_{M,j})$$
$$F_{ij}^{-1} = \frac{adj(F)_{ij}}{det(\mathbf{F})} = \frac{\epsilon_{iPQ}(\delta_{j_2P} + u_{j_2,P})(\delta_{j_3Q} + u_{j_3,Q})}{\epsilon_{LMN}(\delta_{1L} + u_{1,L})(\delta_{2M} + u_{2,M})(\delta_{3N} + u_{3,N})} \tag{5.7}$$

where **S** is the nominal stress, **F** is the deformation Jacobian matrix, **E** is the Green-Lagrange strain, $\vec{u}$ is the deformation vector function, δ_{ij} is the Kronecker delta, and ϵ_{ijk} is the Levi-Civita symbol.

Let $\Pi = U_{strain} - W_{external}$ be the total potential energy of the system. We will consider a virtual displacement $\delta\vec{u}$ due to surface traction and then minimize the first variation of the total potential energy to obtain the body at equilibrium. We will maintain generality by considering unspecified traction vectors $\vec{t^n}$ (the traction at the surface S_t with normal $\hat{n}$). As in the torsional buckling problem we will consider traction over the entire surface (otherwise we require the kinematic boundary condition $\delta\vec{u} = 0$ on all surfaces without traction) and we will neglect body forces. The first variation of the external work is therefore

$$\delta W = \int_{S_t} t_j^n \delta u_j \, dS \tag{5.8}$$

We can construct the first variation of the strain potential energy in material volume V by integrating the first variation of the strain energy density function $\delta\Psi = \frac{\partial\Psi}{\partial\mathbf{E}}\delta\mathbf{E} = \mathbf{S}\mathbf{F}^{-T}\delta\mathbf{E}$

$$\delta U = \int_V \delta\Psi \, dV = \int_V \frac{\partial w}{\partial E_{iM}} F_{jM} F_{jM}^{-1} \delta E_{Mi} \, dV = \int_V S_{ij} F_{Mj}^{-1} \delta E_{Mk} \, dV \tag{5.9}$$

noting that **E** is symmetric and that capital indices are dummy indices. To avoid confusion in notation we will denote the Kronecker delta as γ_{ij}. Using the strain-displacement relation $E_{ij} =$

$\frac{1}{2}(u_{i,j} + u_{j,i} + u_{K,i}u_{K,j})$ we compute the first variation δE_{Mk}

$$\delta E_{Mk} = \frac{\partial E_{Mk}}{\partial u_{Q,R}} \delta u_{Q,R} = \frac{1}{2}(\delta u_{M,k} + \delta u_{k,M} + u_{S,M}\delta u_{S,k} + u_{S,k}\delta u_{S,M}) \tag{5.10}$$

Hence the first variation of the strain energy can be written as follows

$$\delta U = \frac{1}{2}\int_V \frac{\epsilon_{MPQ}(\gamma_{j_2P} + u_{j_2,P})(\gamma_{j_3Q} + u_{j_3,Q})(\delta u_{M,k} + \delta u_{k,M} + u_{S,M}\delta u_{S,k} + u_{S,k}\delta u_{S,M})}{\epsilon_{RST}(\gamma_{1R} + u_{1,R})(\gamma_{2S} + u_{2,S})(\gamma_{3T} + u_{3,T})} S_{ij}\, dV \tag{5.11}$$

Now we wish to define a tensor field $\mathbf{X}(\delta\vec{u})$ with the following divergence

$$\nabla \cdot \mathbf{X}(\delta\vec{u}) = \frac{\epsilon_{MPQ}(\gamma_{j_2P} + u_{j_2,P})(\gamma_{j_3Q} + u_{j_3,Q})(\delta u_{M,k} + \delta u_{k,M} + u_{S,M}\delta u_{S,k} + u_{S,k}\delta u_{S,M})}{2\epsilon_{RST}(\gamma_{1R} + u_{1,R})(\gamma_{2S} + u_{2,S})(\gamma_{3T} + u_{3,T})} \tag{5.12}$$

We can derive a suitable candidate $\mathbf{X}$ by reducing $F^{-1}_{jM}\delta E_{kM}$ and then working backwards from the quotient rule to obtain a first order linear ordinary differential equation with non-constant coefficients. First we have that

$$\mathbf{F}^{-T}\delta\mathbf{E} = \frac{1}{2}\mathbf{F}^{-T}(\delta\mathbf{F}^T\mathbf{F} + \mathbf{F}^T\delta\mathbf{F}) = \frac{1}{2}(\mathbf{F}^{-T}\delta\mathbf{F}^T\mathbf{F} + \delta\mathbf{F}) \tag{5.13}$$

In index notation

$$F^{-1}_{Mj}\delta E_{Mk} = \frac{adj(F)_{Mj}}{det(\mathbf{F})} F_{NM}\delta F_{Nk} + \delta F_{jk} = \frac{adj(F)_{Mj}}{det(\mathbf{F})}(\delta_{NM} + u_{N,M})\delta u_{N,k} + \delta u_{j,k} \tag{5.14}$$

Since $\nabla \cdot \mathbf{X}(\delta\vec{u})$ is a second order tensor we know that $\mathbf{X}$ must be a third rank tensor. We will take the divergence of the third rank tensor in the depth and column vector directions, keeping in line with our convention of taking the divergence of a second rank tensor over the column vectors (i.e. the first column of $\mathbf{F}^{-T}\delta\mathbf{E}$ is from taking the divergence of $\mathbf{X}$ with depth component equal to one, the second with depth component two, etc). Let $d = det(\mathbf{F})$ and consider

$$X_{jkl} = \frac{1}{2}\gamma_{kl}(\int \delta u_{j,l}\, dx_l + \frac{g_{jl}(x_l)}{d(x_l)^{\frac{1}{2}}}) \tag{5.15}$$

where $g_{jl}(x_l)$ is to be determined. Using the quotient rule on $g_{jl}(x_l)d(x_l)^{-\frac{1}{2}}$ we require

$$g_{jl}(x_l)_l d(x_l)^{\frac{1}{2}} - g_{jl}(x_l)d(x_l)^{\frac{1}{2}}_l = adj(F)_{Mj}(\delta_{NM} + u_{N,M})\delta u_{N,l} \tag{5.16}$$

i.e. with respect to variable x_l

$$g'_{jl} - \frac{(\sqrt{d})'}{\sqrt{d}} g_{jl} = \frac{1}{\sqrt{d}} adj(F)_{Mj}(\delta_{NM} + u_{N,M})\delta u_{N,l} \tag{5.17}$$

Hence

$$\begin{aligned} X_{jkl} &= \frac{1}{2}\gamma_{kl} \int \delta u_{j,l}\, dx_l - \frac{1}{2\sqrt{d}}\gamma_{kl} e^{\int \frac{1}{\sqrt{d}}(\sqrt{d})_l\, dx_l} \int \frac{1}{\sqrt{d}} adj(F)_{Mj}(\delta_{NM} + u_{N,M})\delta u_{N,l} e^{-\int \frac{1}{\sqrt{d}}(\sqrt{d})_l\, dx_l}\, dx_l \\ &= \frac{1}{2}\gamma_{kl}\delta u_j - \frac{1}{2}c\gamma_{kl} \int \frac{1}{d} adj(F)_{Mj}(\delta_{NM} + u_{N,M})\delta u_{N,l}\, dx_l \\ &= \frac{1}{2}\gamma_{kl}\delta u_j - \frac{1}{2}c \int \frac{1}{d} adj(F)_{Mj}(\delta_{NM} + u_{N,M})\delta u_{N,k}\, dx_k \end{aligned} \tag{5.18}$$

where c is a constant. Using the product rule we can rewrite (5.10) as

$$\delta U = \int_V (S_{ij}X_{jkl})_i - S_{ij,i}X_{jkl}\, dV \tag{5.19}$$

Now using the divergence theorem

$$\int_V (S_{ij}X_{jkl})_i - S_{ij,i}X_{jkl}\, dV = \int_{S_t} S_{ij}n_iX_{jkl}\, dS - \int_V S_{ij,i}X_{jkl}\, dV \tag{5.20}$$

Substituting δW and δU into $\delta\Pi$ and minimizing the first variation of the total potential energy we obtain

$$
\begin{aligned}
0 &= \int_V S_{ij,i} X_{jkl} \, dV - \int_{S_t} S_{ij} n_i X_{jkl} \, dS + \int_{S_t} t_j^n \delta u_j \, dS \\
&= \int_V S_{ij,i} X_{jkl} \, dV + \int_{S_t} t_j^n \delta u_j - S_{ij} n_i X_{jkl} \, dS
\end{aligned} \tag{5.21}
$$

Splitting up X_{jkl} and equating both sides

$$
\int_V S_{ij,i} \gamma_{kl} \delta u_j \, dV + \int_{S_t} (t_j^n - S_{ij} n_i \gamma_{kl}) \delta u_j \, dS = -\int_V S_{ij,i} \frac{g_{jk}}{\sqrt{d}} \, dV - \int_{S_t} (t_j^n - S_{ij} n_i) \frac{g_{jk}}{\sqrt{d}} \, dS \tag{5.22}
$$

This implies that both sides are zero and that

$$
\int_V S_{ij,i} \delta u_j \, dV + \int_{S_t} (t_j^n - S_{ij} n_i) \delta u_j \, dS = 0 \tag{5.23}
$$

Since δu_j is arbitrary by the fundamental lemma of variational calculus we require that $\nabla \cdot S = S_{ij,i} = 0$ and $(t_k^n - S_{ik} n_i) = 0$. Note that n_i is the undeformed normal vector and hence the second relation forms the traction boundary condition corresponding to the first relation. $\square$

Remark. This proof is confirmation of the minimum potential energy principle. Additionally $\lim_{\delta \vec{u} \to 0} \frac{\partial \mathbf{X}}{\partial x_i} (\delta \vec{u}) = (\lim_{\delta \vec{u} \to 0} F_{ij}^{-1})(\lim_{\delta \vec{u} \to 0} E_{kj}) = (\gamma_{ij})(0) = 0$ such that the divergence of each column/depth tensor field of **X** is 0 under the material volume's original conformation.

5.3 Buckled Potential Energy Functional

In the case of our buckled system we have the following

$$
\Pi = \int_V \Psi_b - (W_{\tilde{T}} + T v_z) - N w_z \, dV - p \int_{S_l} (\rho_i - R_i) + u \, dS_l \tag{5.24}
$$

where u, v, w are only functions of θ and z per the thin-wall assumption, Ψ_b is the buckled strain energy density function, ρ_i is the inner radius of the inflated vessel, S_l is the lateral surface, p is the constant internal pressure, N is the critical axial tension traction, T is the critical torsion traction, $W_{\tilde{T}}$ is an unknown function describing the work due to torsion during loading. Note that $W_{\tilde{T}}$ is independent of the buckling displacement field and that no work is done due to increasing axial tension forces during loading since the stretch ratio $\lambda\Lambda$ is held constant. However, as mentioned in Chapter 3 the stretch ratio assumption is not realistic since the vessel contracts as it twists. Hence, although the vessel is held stationary on the testing apparatus, the vessel is in fact stretching and therefore the axial tension to hold the vessel in place increases and does work on the vessel throughout torsional loading.

If we define the potential energy of the buckled vessel relative to the last stable state and pull the work due to pressure into the volumetric integral then we have the following

$$\Pi = \int_V \Psi_b - \Psi_{pb} - \frac{2\rho_i}{\rho_o^2 - \rho_i^2} pu - Tv_z - Nw_z \, dV \tag{5.25}$$

Note that this simplification occurs automatically upon taking the first variation. We will choose the pre-buckling state to be the point at which the loading of the torsion and axial tension become constant, but we will not make assumptions about whether it is just before or on the cusp of instability. The state is predominantly defined by the angle of twist γ and will be close to buckling. Otherwise, choosing a small angle of twist γ means we must make strong assumptions about the loading curves of T and N.

There are three types of equilibria in the context of mechanical failure. The first is the stable equilibrium which, by the minimum potential energy principle, dictates that an object subjected to subcritical loads naturally deforms into the lowest energy configuration. In this state if an applied force is removed from the system then the system deforms back to its original configuration. The second is a neutral equilibrium in which an object which is perturbed by an applied force will maintain the perturbed deformation when the applied force is removed. It is essentially the cusp of mechanical failure and is identified by a stationary inflection point in Π. At the neutral equilibrium point the vessel may bifurcate from a state of stability to instability or maintain stability and is therefore sought after if the objective is to obtain a relation between critical loading parameters.

The third is the unstable equilibrium in which an object which is perturbed slightly experiences large changes in deformation.

In general we must look at higher dimensional variations to classify critical points. Broadly speaking, $\delta^2\Pi_{pb} > 0$ indicates a minimum (stable) for $\delta\Pi$, $\delta^2\Pi_{pb} < 0$ indicates a maximum (unstable), and $\delta^2\Pi = 0$ indicates an inflection point (neutral). Any critical point for Π automatically satisfies the equilibrium equations, as we already proved, however we specifically want either the maximization of the total energy functional or a neutral equilibrium. If we are to seek buckling displacement configurations then we wish to maximize the total potential energy functional to obtain an unstable displacement configuration. If we seek a relation between critical loads then we wish to obtain a stationary inflection point so that the object is in neutral equilibrium and is therefore on the cusp of bifurcating into an unstable configuration. In either case we seek to minimize the first variation of the total potential energy functional. Defining the pre-buckling state as we have above implies that trivial solutions which do not impact the total potential energy functional will not push the vessel into a range of instability but will instead maintain a neutral or stable equilibrium.

5.4 First Variation of Buckled System (First-Order Q)

The Fung strain-energy density function has the form $\Psi = \frac{1}{2}C_0(e^Q - 1)$ where Q is given by

$$Q = b_1 E_{RR}^2 + b_2 E_{\Theta\Theta}^2 + b_3 E_{ZZ}^2 + 2b_4 E_{RR}E_{\Theta\Theta} + 2b_5 E_{\Theta\Theta}E_{ZZ} + 2b_6 E_{ZZ}E_{RR} + b_7(E_{R\Theta}^2 + E_{\Theta R}^2) + b_8(E_{\Theta Z}^2 + E_{Z\Theta}^2) + b_9(E_{RZ}^2 + E_{ZR}^2) \tag{5.26}$$

where each E_{ij} is the usual Green-Lagrange strain. Since the vessel geometry is assumed to be a cylindrical shell, the undeformed radius R and deformed radius ρ become constants. The Fung strain-energy density for the radially, circumferentially, and axially loaded pre-buckling cylindrical shell can therefore be written

$$\Psi_{pb} = \frac{1}{2}C_0(e^{C_7} - 1) \tag{5.27}$$

where C_7 is a constant. If we consider the buckled vessel configuration mappings and approximate Q_b to the first-order we obtain the following form

$$\Psi_b = \frac{1}{2}C_0(e^{C_1+C_2u+C_3\frac{\partial v}{\partial\theta}+C_4\frac{\partial v}{\partial z}+C_5\frac{\partial w}{\partial\theta}+C_6\frac{\partial w}{\partial z}} - 1) \tag{5.28}$$

where u, v, w are the displacements in the radial, circumferential, and axial directions upon buckling and the rest of the terms are constants. The constants are derived by computing the Green-Lagrange strain components and substituting into the Fung strain energy density function. After simplifying terms which are necessarily zero (in this loading case) we have the form seen above.

In infinitesimal strain theory quadratic displacement terms are neglected in the strain-displacement relations $e_{ij} = \frac{1}{2}(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i})$ where $U = (u, v, w)$. If we make the assumption that the displacements are small, we can expand the strain energy function as a power series and reduce the function to a first-order approximation of the exponential.

$$\Psi_b = \frac{1}{2}C_0[e^{C_1}(1 + C_2u + C_3\frac{\partial v}{\partial\theta} + C_4\frac{\partial v}{\partial z} + C_5\frac{\partial w}{\partial\theta} + C_6\frac{\partial w}{\partial z}) - 1] \tag{5.29}$$

This approach is inadequate for variational calculus as the Euler-Lagrange equations are largely uninformative.

In finite-strain theory deformable bodies experience large deformations such that quadratic terms become important. The strain-displacement equations in large-deformation theory become $E_{ij} = \frac{1}{2}(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} + \frac{\partial U_M}{\partial x_i}\frac{\partial U_M}{\partial x_j})$. Hyperelastic materials (those whose material properties are defined using a strain energy density function) necessarily experience large deformations so a second-order approximation is more appropriate. If we maintain the first-order estimate of the buckled Q_b and extend the approximation of the exponential to allow second-order terms we have the following

$$
\begin{aligned}
e^{-C_1}(\Psi_b \frac{2}{C_0} + 1) = \frac{3}{2} &+ C_2 u(1 + \frac{1}{2}C_2 u + C_3 v_\theta + C_4 v_z + C_5 w_\theta + C_6 w_z) \\
&+ C_3 v_\theta(1 + \frac{1}{2}C_3 v_\theta + C_4 v_z + C_5 w_\theta + C_6 w_z) \\
&+ C_4 v_z(1 + \frac{1}{2}C_4 v_z + C_5 w_\theta + C_6 w_z) + C_5 w_\theta(1 + \frac{1}{2}C_5 w_\theta + C_6 w_z) \\
&+ C_6 w_z(1 + \frac{1}{2}C_6 w_z)
\end{aligned}
\tag{5.30}
$$

where of course the accuracy is limited to the first-order assumptions made in deriving the original Q_b.

Proceeding with the first-order estimation of Q_b and second-order approximation of the exponential, computing the first variation $\delta\Pi$ we have the following

$$
\delta\Pi = \int_V \delta(\Delta\Psi) - \delta(\sum_i W_i)\, dV = \int_V \delta\Psi_b - \sum_i \delta W_i\, dV \tag{5.31}
$$

For $\delta\Psi_b$ we have

$$
\begin{aligned}
\delta\Psi_b \frac{2}{C_0} e^{-C_1} = C_2[&\delta u + C_2 u\delta u + C_3(u\delta v_\theta + v_\theta\delta u) + C_4(u\delta v_z + v_z\delta u) + C_5(u\delta w_\theta + w_\theta\delta u) \\
&+ C_6(u\delta w_z + w_z\delta u)] + C_3[\delta v_\theta + C_3 v_\theta\delta v_\theta + C_4(v_\theta\delta v_z + v_z\delta v_\theta) + C_5(v_\theta\delta w_\theta + w_\theta\delta v_\theta) \\
&+ C_6(v_\theta\delta w_z + w_z\delta v_\theta)] + C_4[\delta v_z + C_4 v_z\delta v_z + C_5(v_z\delta w_\theta + w_\theta\delta v_z) \\
&+ C_6(v_z\delta w_z + w_z\delta v_z)] + C_5[\delta w_\theta + C_5 w_\theta\delta w_\theta + C_6(w_\theta\delta w_z + w_z\delta w_\theta)] \\
&+ C_6(\delta w_z + C_6 w_z\delta w_z)]
\end{aligned}
\tag{5.32}
$$

and for $\sum_i \delta W_i$

$$
\sum_i \delta W_i = -\frac{2\rho_i}{\rho_o^2 - \rho_i^2} p\delta u - T\delta v_z - N\delta w_z \tag{5.33}
$$

5.4.1 Euler-Lagrange Equations

To find functions which extremize the total potential energy we set $\delta\Pi = 0$. After integrating by parts and since $\delta u, \delta v, \delta w$ are independent and arbitrary by the fundamental lemma of variational calculus we obtain the following Euler-Lagrange equations

$$0 = \frac{1}{2}C_0 e^{C_1} C_2 - p\frac{2\rho_i}{\rho_o^2 - \rho_i^2} + \frac{1}{2}C_1 e^{C_1} C_2 (C_2 u + C_3 v_\theta + C_4 v_z + C_5 w_\theta + C_6 w_z) \tag{5.34}$$

$$0 = -C_3(C_2 u_\theta + C_3 v_{\theta\theta} + C_4 v_{z\theta} + C_5 w_{\theta\theta} + C_6 w_{z\theta}) - C_4(C_2 u_z + C_3 v_{\theta z} + C_4 v_{zz} + C_5 w_{\theta z} + C_6 w_{zz})$$

$$0 = -C_5(C_2 u_\theta + C_3 v_{\theta\theta} + C_4 v_{z\theta} + C_5 w_{\theta\theta} + C_6 w_{z\theta}) - C_6(C_2 u_z + C_3 v_{\theta z} + C_4 v_{zz} + C_5 w_{\theta z} + C_6 w_{zz})$$

Since the ends of the vessel are fixed the above equations and essential boundary conditions $u = v = w = 0$ at $Z = 0, L_0$ form a Dirichlet boundary-value problem.

Immediately we see that $\frac{C_3}{C_4} = \frac{C_5}{C_6}$ and we require that $p = \frac{1}{4}(\rho_o^2 - \rho_i^2)C_0 e^{C_1} C_2$. This equation for pressure is the only nontrivial equation if we were to obtain the Euler-Lagrange equations using the first-order approximation of the exponential. Equating the bottom two equations using u_z we obtain

$$\begin{aligned} &\frac{1}{C_4}[C_3(C_2 u_\theta + C_3 v_{\theta\theta} + C_4 v_{z\theta} + C_5 w_{\theta\theta} + C_6 w_{z\theta}) + C_4(C_3 v_{\theta z} + C_4 v_{zz} + C_5 w_{\theta z} + C_6 w_{zz})] \\ =&\frac{1}{C_6}[C_5(C_2 u_\theta + C_3 v_{\theta\theta} + C_4 v_{z\theta} + C_5 w_{\theta\theta} + C_6 w_{z\theta}) + C_6(C_3 v_{\theta z} + C_4 v_{zz} + C_5 w_{\theta z} + C_6 w_{zz})] \end{aligned} \tag{5.35}$$

Looking at the second and mixed derivative operators (before substituting for C_3 which zeroes the equation) we see a consistent correspondence between v and w, namely $C_4 v = C_6 w$. For example

$$\frac{1}{C_4}\frac{\partial^2}{\partial\theta^2}(C_3 C_5 w + \frac{C_4^2 C_5^2}{C_6^2} v) \sim \frac{1}{C_6}\frac{\partial^2}{\partial\theta^2}(C_3 C_5 v + C_5^2 w) \tag{5.36}$$

5.4.2 Change of Variables

After substituting for w into the second Euler-Lagrange equation we obtain

$$\begin{aligned} 0 &= C_3(C_2u_\theta + (C_3 + \frac{C_4C_5}{C_6})v_{\theta\theta} + 2C_4v_{z\theta}) + C_4(C_2u_z + (C_3 + \frac{C_4C_5}{C_6})v_{\theta z} + 2C_4v_{zz}) \\ &= C_3(C_2u_\theta + 2C_3v_{\theta\theta} + 2C_4v_{z\theta}) + C_4(C_2u_z + 2C_3v_{\theta z} + 2C_4v_{zz}) \\ &= \frac{1}{2}C_2(C_3u_\theta + C_4u_z) + C_3^2v_{\theta\theta} + 2C_3C_4v_{z\theta} + C_4^2v_{zz} \end{aligned} \tag{5.37}$$

Substituting for u from the first Euler-Lagrange equation yields a trivial equation. Apply the following change of variables

$$u = \frac{2}{C_2}u' \qquad x = \frac{1}{C_3}\theta \qquad y = \frac{1}{C_4}z \tag{5.38}$$

The equation is then

$$u'_x + u'_y + v_{xx} + v_{yy} + 2v_{xy} = 0 \tag{5.39}$$

Let us assume that $\nabla \cdot u = 0$ everywhere under the original coordinate system. If it holds within the vessel it will also hold everywhere. Then we require that

$$u_\theta = -ru_z \tag{5.40}$$

Using the chain rule and the above relation

$$\begin{aligned} u'_x &= \frac{1}{2}C_2C_3(-ru_z) \\ u'_y &= \frac{1}{2}C_2C_4u_z \end{aligned} \tag{5.41}$$

Choosing $r = \frac{C_4}{C_3}$we obtain

$$v_{xx} + v_{yy} + 2v_{xy} = 0 \tag{5.42}$$

which both v and w must satisfy. Note the following algebraic similarity

$$v_{xx} + v_{yy} + 2v_{xy} \sim X^2 + Y^2 + 2XY = (X + Y)^2 \tag{5.43}$$

If we apply the seemingly logical change of variable $\xi = x + y$ then we have $v_{\xi\xi} = 0$. This is a trivial solution to the Euler-Lagrange equations in the sense that it contributes only constant terms to the total potential energy functional. The result is a maintenance of the neutral or stable equilibrium as the constant terms must cancel thereby negating any contribution to the energy functional from the external forces. If we instead apply the following change of variables

$$\xi = \frac{1}{2}(x + y) \qquad\qquad \eta = y \tag{5.44}$$

we obtain the following partial differential equation

$$v_{\xi\xi} + v_{\eta\eta} = 0 \tag{5.45}$$

which is recognizable as Laplace's equation with a slightly altered geometric interpretation. Since $w = \frac{C_4}{C_6}v$ we also require

$$w_{\xi\xi} + w_{\eta\eta} = 0 \tag{5.46}$$

Now if we examine the first Euler-Lagrange equation and substitute for w we have the following expression for u in terms of v

$$u = -\frac{2}{C_2}(C_3 v_\theta + C_4 v_z) \tag{5.47}$$

The nontrivial solutions under this change of variable will map the vessel from a stable equilibrium (unbuckled state) to a neutral or unstable equilibrium. Checking the second variation will determine if these solutions maximize the total potential energy functional thereby giving us buckling displacement configurations.

5.5 First Variation of Buckled System (Second-Order Q)

The second-order approximation to Q is used to illustrate the increasing complications that develop for marginal gains in accuracy. Proceeding in a similar fashion as above, the second-order approximation of Q_b gives us the following

$$\begin{aligned} Q_b = C_1 &+ u(D_1 + D_2 u + D_3 u_\theta + D_4 u_z + D_5 v_\theta + D_6 v_z + D_7 w_\theta + D_8 w_z) \\ &+ u_\theta(D_9 + D_{10} u_\theta + D_{11} u_z + D_{12} v_\theta + D_{13} v_z + D_{14} w_\theta + D_{15} w_z) \\ &+ u_z(D_{16} + D_{17} u_z + D_{18} v_\theta + D_{19} v_z + D_{20} w_\theta + D_{21} w_z) \\ &+ v_\theta(D_{22} + D_{23} v_\theta + D_{24} v_z + D_{25} w_\theta + D_{26} w_z) \\ &+ v_z(D_{27} + D_{28} v_z + D_{29} w_\theta + D_{30} w_z) + w_\theta(D_{31} + D_{32} w_\theta + D_{33} w_z) \\ &+ w_z(D_{34} + D_{35} w_z) \end{aligned} \tag{5.48}$$

where D_i are constants. Then for e^Q we have the following

$$
\begin{aligned}
e^Q e^{-C_1} = 1 &+ u[D_1 + (D_2 + \frac{1}{2}D_1^2)u + (D_3 + D_1D_9)u_\theta + (D_4 + D_1D_{16})u_z + (D_5 + D_1D_{22})v_\theta \\
&+ (D_6 + D_1D_{27})v_z + (D_7 + D_1D_{31})w_\theta + (D_8 + D_1D_{34}w_z)] + u_\theta[D_9 + (D_{10} + \frac{1}{2}D_9^2)u_\theta \\
&+ (D_{11} + D_9D_{16})u_z + (D_{12} + D_9D_{22})v_\theta + (D_{13} + D_9D_{27})v_z + (D_{14} + D_9D_{31})w_\theta \\
&+ (D_{15} + D_9D_{34})w_z] + u_z[D_{16} + (D_{17} + \frac{1}{2}D_{16}^2)u_z + (D_{18} + D_{16}D_{22})v_\theta + (D_{19} + D_{16}D_{27})v_z \\
&+ (D_{20} + D_{16}D_{31})w_\theta + (D_{21} + D_{16}D_{34})w_z] + v_\theta[D_{22} + (D_{23} + \frac{1}{2}D_{22}^2)v_\theta \qquad (5.49) \\
&+ (D_{24} + D_{22}D_{27})v_z + (D_{25} + D_{22}D_{31})w_\theta + (D_{26} + D_{22}D_{34})w_z) + v_z[D_{27} + (D_{28} + \frac{1}{2}D_{27}^2)v_z \\
&+ (D_{29} + D_{27}D_{31})w_\theta + (D_{30} + D_{27}D_{34})w_z] + w_\theta[D_{31} + (D_{32} + \frac{1}{2}D_{31}^2)w_\theta \\
&+ (D_{33} + D_{27}D_{34})w_z] + w_z[D_{34} + (D_{35} + \frac{1}{2}D_{34}^2)w_z]
\end{aligned}
$$

We will omit the formal expression for $\delta\Psi_b$ since it is lengthy and proceeds in an identical manner to equation (5.32). The first variation of the work term is identical to that in Section 5.4.

5.5.1 Euler-Lagrange Equations

After omitting $p = \frac{1}{4}(\rho_o^2 - \rho_i^2)C_0e^{C_1}D_1$ (note that $D_1 = C_2$ from Section 5.4.1) we have the following three Euler-Lagrange equations

$$
\begin{aligned}
0 = (2D_2 + D_1^2)u &+ (D_3 + D_1D_9)u_\theta + (D_4 + D_1D_{16})u_z + (D_5 + D_1D_{22})v_\theta + (D_6 + D_1D_{27})v_z \\
&+ (D_7 + D_1D_{31})w_\theta + (D_8 + D_1D_{34})w_z - [(D_3 + D_1D_9)u_\theta + (2D_{10} + D_9^2)u_{\theta\theta} \\
&+ (D_{11} + D_9D_{16})u_{z\theta} + (D_{12} + D_9D_{22})v_{\theta\theta} + (D_{13} + D_9D_{27})v_{z\theta} \qquad (5.50) \\
&+ (D_{14} + D_9D_{31})w_\theta + (D_{15} + D_9D_{34})w_z] - [(D_{11} + D_9D_{16})u_{\theta z} + (2D_{17} + D_{16}^2)u_z \\
&+ (D_{18} + D_{16}D_{22})v_{\theta z} + (D_{19} + D_{16}D_{27})v_{zz} + (D_{20} + D_{16}D_{31})w_{\theta z} + (D_{21} + D_{16}D_{34})w_{zz}]
\end{aligned}
$$

$$
\begin{aligned}
0 &= (D_5 + D_1 D_{22})u_\theta + (D_{12} + D_9 D_{22})u_{\theta\theta} + (D_{18} + D_{16}D_{22})u_{z\theta} + (2D_{23} + D_{22}^2)v_{\theta\theta} \\
&\quad + (D_{24} + D_{22}D_{27})v_{z\theta} + (D_{25} + D_{22}D_{31})w_{\theta\theta} + (D_{26} + D_{22}D_{34})w_{z\theta} \\
&\quad + (D_6 + D_1 D_{27})u_z + (D_{13} + D_9 D_{27})u_{\theta z} + (D_{19} + D_{16}D_{27})u_{zz} + (D_{24} + D_{22}D_{27})v_{\theta z} \\
&\quad + (2D_{28} + D_{27}^2)v_{zz} + (D_{29} + D_{27}D_{31})w_{\theta z} + (D_{30} + D_{27}D_{34})w_{zz}
\end{aligned} \tag{5.51}
$$

$$
\begin{aligned}
0 &= (D_7 + D_1 D_{31})u_\theta + (D_{14} + D_9 D_{31})u_{\theta\theta} + (D_{20} + D_{16}D_{31})u_{z\theta} + (D_{25} + D_{22}D_{34})v_{\theta\theta} \\
&\quad + (D_{29} + D_{27}D_{31})v_{z\theta} + (2D_{32} + D_{31}^2)w_{\theta\theta} + (D_{33} + D_{27}D_{34})w_{z\theta} \\
&\quad + (D_8 + D_1 D_{34})u_z + (D_{15} + D_9 D_{34})u_{\theta z} + (D_{21} + D_{16}D_{34})u_{zz} + (D_{26} + D_{22}D_{34})v_{\theta z} \\
&\quad + (D_{30} + D_{27}D_{34})v_{zz} + (D_{33} + D_{27}D_{34})w_{\theta z} + (2D_{35} + D_{34}^2)w_{zz}
\end{aligned} \tag{5.52}
$$

Since the ends of the vessel are fixed the above equations and essential boundary conditions $u = v = w = 0$ at $Z = 0, L_0$ form a Dirichlet boundary-value problem.

Comparing the last two equations gives us correspondences between the coefficients of the partial derivatives. Furthermore, if we are to derive the second-order Q from the buckled configuration mappings we can eliminate several coefficients outright since they will be zero. After substituting and simplifying, the first Euler-Lagrange equation becomes the following

$$
\begin{aligned}
0 &= (2D_2 + D_1^2)u - 2D_{10}u_{\theta\theta} - 2D_{11}u_{z\theta} - 2D_{17}u_{zz} + (D_5 + D_1 D_{22})(v_\theta + w_\theta) \\
&\quad + (D_6 + D_1 D_{27})(v_z + w_z)
\end{aligned} \tag{5.53}
$$

Noting that we have used the follow relations from the original equations

$$
\begin{aligned}
(D_5 + D_1 D_{22}) &= (D_7 + D_1 D_{31}) \\
(D_6 + D_1 D_{27}) &= (D_8 + D_1 D_{34})
\end{aligned} \tag{5.54}
$$

The second Euler-Lagrange equation becomes

$$\begin{aligned} 0 &= (D_5 + D_1 D_{22})u_\theta + (D_6 + D_1 D_{27})u_z + (2D_{23} + D_{22}^2)(v_{\theta\theta} + w_{\theta\theta}) \\ &+ (D_{24} + D_{22}D_{27})(v_{z\theta} + w_{z\theta}) + (2D_{28} + D_{27}^2)(v_{zz} + w_{zz}) \end{aligned} \tag{5.55}$$

Noting that similarly we have used that

$$\begin{aligned} (2D_{23} + D_{22}^2) &= (D_{25} + D_{22}D_{31}) \\ (D_{24} + D_{22}D_{27}) &= (D_{29} + D_{27}D_{31}) \\ (2D_{28} + D_{27}^2) &= (D_{30} + D_{27}D_{34}) \end{aligned} \tag{5.56}$$

As before assume $\nabla \cdot \vec{u} = 0$ and choose $r = \frac{D_6 + D_1 D_{27}}{D_5 + D_1 D_{22}}$ so that equation (5.55) becomes

$$0 = (2D_{23} + D_{22}^2)(v_{\theta\theta} + w_{\theta\theta}) + (D_{24} + D_{22}D_{27})(v_{z\theta} + w_{z\theta}) + (2D_{28} + D_{27}^2)(v_{zz} + w_{zz}) \tag{5.57}$$

As expected, using a more accurate approximation of Q results in increasingly complicated partial differential equations. Interestingly the summation of the buckling displacements in the axial and circumferential directions seem to satisfy a similar equation as that derived in Section 5.4. We could use a change of variables to simplify the system but without explicit expressions for each coefficient we would need to divide the substitution into separate cases based on the discriminant. However, even solving for $v + w$ we would still have a nonhomogeneous partial differential equation for u from the first Euler-Lagrange equation which would require some considerable effort to solve, unlike in Section 5.4. Solving both sets of partial differential equations numerically and comparing would be an easy method of gauging whether a first-order estimation of Q is sufficient so that analytical solutions to equations (5.53) and (5.57) are unnecessary.

Chapter 6: APPROXIMATION OF FIRST VARIATION

6.1 Talyor Polynomial of Π

Let us define a new gradient $\nabla^{\delta\vec{u}}$ and $\nabla^{\delta\vec{u}'}$ for variational calculus

$$\begin{aligned}\nabla^{\delta\vec{u}} &:= < \frac{\partial}{\partial u}, \frac{\partial}{\partial v}, \frac{\partial}{\partial w} > \\ \nabla^{\delta\vec{u}'} &:= < \frac{\partial}{\partial u'}, \frac{\partial}{\partial v'}, \frac{\partial}{\partial w'} >\end{aligned} \tag{6.1}$$

The δ is included to differentiate from the directional derivative. Suppose the pre-buckling total potential energy $\Pi_{pb} = \Pi_{pb}(u, u', v, v', w, w')$ so that the buckled configuration mappings are given by

$$\begin{aligned}\rho' &= u + \delta u \\ \phi' &= v + \delta v \\ \zeta' &= w + \delta w\end{aligned} \tag{6.2}$$

where $\delta\vec{u}$ is the first variation of $\vec{u}$. Note that u, v, w are different in this approximation than when evaluating the first variation of Π_b although they are named the same. Unlike when we minimized the first variation of the system, we will treat the pre-buckling state to be precisely the neutral equilibrium so that any nontrivial perturbations map the pre-buckling state to the buckled state. Constructing the variational Taylor series of Π we can take the difference of the perturbed state $\delta\vec{u}$

with the stable state to approximate the first and second variation.

$$
\begin{aligned}
\Delta\Pi &= U_b - U_{pb} - W_{pressure} - W_{torque} - W_{tension} \\
&\approx \nabla^{\delta\vec{u}}\Pi_{pb}\cdot\delta\vec{u} + \nabla^{\delta\vec{u'}}\Pi_{pb}\cdot\delta\vec{u'} + \frac{1}{2}\delta\vec{u}\cdot\nabla^{\delta\vec{u}}\nabla^{\delta\vec{u}}\Pi_{pb}\cdot\delta\vec{u} + \delta\vec{u}\cdot\nabla^{\delta\vec{u}}\nabla^{\delta\vec{u'}}\Pi_{pb}\cdot\delta\vec{u'} \\
&\quad + \frac{1}{2}\delta\vec{u'}\cdot\nabla^{\delta\vec{u'}}\nabla^{\delta\vec{u'}}\Pi_{pb}\cdot\delta\vec{u'} \\
&= \delta\Pi_{pb} + \delta^2\Pi_{pb}
\end{aligned}
\tag{6.3}
$$

where U_k is the total strain energy of a particular state k and W_l is external work due to a traction l.

By setting $\Delta\Pi := 0$ we obtain an energy conservation equation which approximates a stationary inflection point (therefore satisfying equilibrium). In this case $\delta^2\Pi \approx 0$ which should imply that the vessel is on the cusp of instability. However, note that the pre-buckling state is by definition the bifurcation point and therefore the neutral equilibrium. Hence any nontrivial buckling configurations will therefore result in $\Pi \leq 0$ as the work terms begin to outweigh the strain energy terms as expected during mechanical failure. In our case we are in fact seeking deformations which maintain the neutral equilibrium so we may identify a relation between critical loading parameters. However, although we seek deformations which are trivial in contribution to Π, we seek nontrivial relations between the displacements and the buckling modes.

6.2 $\Delta\Pi$

Integrating the strain energy density function of one configuration state over its domain yields the total strain energy of the configuration's particular deformation pattern. Integrating over their acting surfaces the product of the external traction with their corresponding caused displacement gives the external work W_f of each applied load. As we have shown, taking the difference of two states very close together can be used to approximate the first variation. Considering the initial state to be the moment just prior to buckling and the final state the perturbed stable state (and hence start of mechanical failure) allows us to write $\delta W_{ext} \approx W_f$. and $\delta U_{strain} \approx U_b - U_{pb}$.

Computation of the strain energy of the pre-buckled state is trivial since its strain energy density function is constant throughout the body. However, the buckled strain energy density function will necessarily depend on the buckling displacement field. Buckling displacement fields are typically trigonometric, and as we showed in our previous work "Solutions to the First-Order Buckling Equations of a Fung Hyperelastic Cylindrical Shell Subjected to Torsion, Internal Pressure, and Axial Tension" in satisfaction of the Master of Science degree in biomedical engineering, the assumptions made in this problem imply single-order trigonometric buckling displacements. As such, integrating over the material volume, and hence their periods, will be zero if we suppose a first-order deformation pattern. This keeps in line with our requirement that the total potential energy functional does not change so that we maintain the bifurcation point. However it does not tell us much about the relation of eigenmodes and material parameters.

We can instead consider the ratio by which the strain energy changes per change in external work. As will be proven in Proposition 6.2.1, since the frequencies of the three displacement functions are assumed equal and amplitudes constant per our previous work, we can simply integrate over a fraction of the total period 2π knowing that the ratio is kept in tact (assuming the same integration constraint is applied everywhere). If we let $< \Omega, \Phi, \eta >$ be the displacement field that maps the pre-buckling state to the buckled state, the energy equation is as follows

$$0 = \int_0^{2\pi} \int_0^{L_0} \int_{\rho_i}^{\rho_o} \Psi_b - \Psi_{pb} \, R \, dR \, dZ \, d\Theta - p\rho_i \int_0^{2\pi} \int_0^{L_0} \Omega \, dZ \, d\Theta - T \int_0^{2\pi} \int_{\rho_i}^{\rho_o} \Phi \, R \, dR \, d\Theta \Big|_0^{L_0} - N \int_0^{2\pi} \int_{\rho_i}^{\rho_o} \eta \, R \, dR \, d\Theta \Big|_0^{L_0} \tag{6.4}$$

Remark. We can add factor of $\frac{1}{2}$ in front of the torsional work term if we assume the torsional loading upon buckling is not constant but instead linearly proportional to the angular displacement. Since T presented here is the torsional traction upon the moment buckling occurs and since the torsional work is the area under the $\tilde{T}(\Phi)$ curve (where $\tilde{T}$ is an "apparent" torsional traction), the torsional work should be half of the value as that when the torsional loading is assumed constant.

Proposition 6.2.1. *If the pre-buckling state of a hyperelastic thin-wall cylinder is perturbed by trigonometric displacements which share the same frequency and have constant amplitudes, then the ratio of change in work to change in strain energy can be represented by an energy balance on a small volumetric slice of the cylinder.*

Proof. Suppose

$$\Omega = A\sin(\frac{m\pi Z}{L_0} + n\Theta) \qquad \Phi = B\cos(\frac{m\pi Z}{L_0} + n\Theta) \qquad \eta = C\cos(\frac{m\pi Z}{L_0} + n\Theta) \tag{6.5}$$

The perturbations change sign symmetrically with respect to the pre-buckling configuration by construction, but if we take the absolute value of the displacements and integrate we obtain the total change in energy. To guarantee positivity for both trigonometric functions we will integrate over the first quarter-wave in the circumferential and height directions and then multiply the integration by the total number of quarter-waves to obtain the total distance. This distance will then be used to compute the absolute value of the buckled strain energy and the absolute value of the external work.

By hypothesis the number of quarter-waves in the circumferential and height directions are $4n$ and $2m$ respectively. Carrying out the integration and using that Ψ_{pb} is constant we have

$$\begin{aligned} 0 = 8nm \int_0^{\frac{\pi}{2n}} \int_0^{\frac{L_0}{2m}} \int_{\rho_i}^{\rho_o} \Psi_b \, R \, dR \, dZ \, d\Theta - \int_0^{2\pi} \int_0^{L_0} \int_{\rho_i}^{\rho_o} \Psi_{pb} \, R \, dR \, dZ \, d\Theta \\ - 8nmp\rho_i \int_0^{\frac{\pi}{2n}} \int_0^{\frac{L_0}{2m}} \Omega \, dZ \, d\Theta - 8nm \int_0^{\frac{\pi}{2n}} \int_{\rho_i}^{\rho_o} T\Phi + N\eta \, R \, dR \, d\Theta|_0^{\frac{L_0}{2m}} \end{aligned}$$

$$\begin{aligned} = \int_0^{\frac{\pi}{2n}} \int_0^{\frac{L_0}{2m}} \int_{\rho_i}^{\rho_o} \Psi_b \, R \, dR \, dZ \, d\Theta - \frac{1}{8nm} \int_0^{2\pi} \int_0^{L_0} \int_{\rho_i}^{\rho_o} \Psi_{pb} \, R \, dR \, dZ \, d\Theta \\ - p\rho_i \int_0^{\frac{\pi}{2n}} \int_0^{\frac{L_0}{2m}} \Omega \, dZ \, d\Theta - \int_0^{\frac{\pi}{2n}} \int_{\rho_i}^{\rho_o} T\Phi + N\eta \, R \, dR \, d\Theta|_0^{\frac{L_0}{2m}} \end{aligned}$$

$$= \int_0^{\frac{\pi}{2n}} \int_0^{\frac{L_0}{2m}} \int_{\rho_i}^{\rho_o} \Psi_b \, R \, dR \, dZ \, d\Theta - \int_0^{\frac{\pi}{2n}} \int_0^{\frac{L_0}{2m}} \int_{\rho_i}^{\rho_o} \Psi_{pb} \, R \, dR \, dZ \, d\Theta$$
$$- p\rho_i \int_0^{\frac{\pi}{2n}} \int_0^{\frac{L_0}{2m}} \Omega \, dZ \, d\Theta - \int_0^{\frac{\pi}{2n}} \int_{\rho_i}^{\rho_o} T\Phi + N\eta \, R \, dR \, d\Theta|_0^{\frac{L_0}{2m}} \tag{6.6}$$

which is equivalent to the energy balance taken over a volumetric sliver of the material volume, as was to be proved. □

Remark. Note that the traction terms are not scaled since they are already normalized by surface area.

Corollary 6.2.1.1. $\Delta\Pi = 0$ *must hold over any volumetric sliver extending across the thickness of the vessel.*

Proof. Consider the contradiction if this were not true but the entire volume of the material were this volumetric slice. □

Using the results above we can begin to examine arbitrary slices. If we take the energy balance over the first half-wave of the height and the first half-wave of the circumferential direction simultaneously, the result is the same as taking the integration over the whole material volume (trivial equation). If we look only at the energy balance on a volumetric sliver extending from ρ_i to ρ_o, 0 to $\frac{\pi}{n}$, and 0 to L_0 in the radial, circumferential, and z-directions respectively the result is an equation that becomes trivial whenever $m \in \mathbb{N}$. Since m is an integer by assumption, the subsequent equation is not useful. We seek an equation which is not trivial somewhere in the acceptable domain for eigenvalues m and n. Consider shortening the height of the sliver to $\frac{L_0}{2m}$ (the first quarter-wave in the z-direction). We obtain the following energy equation

$$0 = \int_0^{\frac{\pi}{n}} \int_0^{\frac{L_0}{2m}} \int_{\rho_i}^{\rho_o} \frac{1}{2} C_0 [e^{C_1}(1 + C_2\Omega + C_3\frac{\partial \Phi}{\partial \Theta} + C_4\frac{\partial \Phi}{\partial Z} + C_5\frac{\partial \eta}{\partial \Theta} + C_6\frac{\partial \eta}{\partial Z}) - 1] \, R \, dR \, dZ \, d\Theta$$
$$- \int_0^{\frac{\pi}{n}} \int_0^{\frac{L_0}{2m}} \int_{\rho_i}^{\rho_o} \frac{1}{2} C_0 (e^{C_7} - 1) \, R \, dR \, dZ \, d\Theta - p\rho_i \int_0^{\frac{\pi}{n}} \int_0^{\frac{L_0}{2m}} \Omega \, dZ \, d\Theta - T \int_0^{\frac{\pi}{n}} \int_{\rho_i}^{\rho_o} \Phi \, R \, dR \, d\Theta|_0^{\frac{L_0}{2m}}$$
$$- N \int_0^{\frac{\pi}{n}} \int_{\rho_i}^{\rho_o} \eta \, R \, dR \, d\Theta|_0^{\frac{L_0}{2m}} \tag{6.7}$$

We must make assumptions regarding the buckled displacement field in order to compute these integrals. By inspection (as will be shown later) the linearized partial differential equations of static equilibrium permit trigonometric solutions. We may therefore assume the displacements can be represented as 2D Fourier sine and cosine series

$$\begin{aligned} \Omega &= \sum_{n,m=1}^{\infty} a_{mn} \sin(\frac{m\pi Z}{L_0} + n\Theta) \\ \Phi &= \sum_{n,m=0}^{\infty} b_{mn} \cos(\frac{m\pi Z}{L_0} + n\Theta) \\ \eta &= \sum_{n,m=0}^{\infty} c_{mn} \cos(\frac{m\pi Z}{L_0} + n\Theta) \end{aligned} \tag{6.8}$$

For linear elastic materials it is a valid assumption that if the vessel is sufficiently long then the ends of the vessel have little impact on the buckling conformation and can therefore be ignored [24]. This permits single Fourier term solutions like those seen in Proposition 6.2.1. If we evaluate the integration assuming single term trigonometric displacements we obtain similar results to the assumption that the displacements are Fourier sine and cosine series. Instead of relating A, B, C we would have sums of Fourier coefficients, but as long as we are consistent in applying the same assumption to the static equilibrium the problem is very similar. It is identically the same problem if we can assume that each term in the series of each component is itself a solution and can be paired with the other components term-by-term. In this case instead of A, B, C we will have Fourier coefficient a_{mn}, b_{mn}, c_{mn} and subsequently infinitely many equations.

The Fourier series assumption has the potential to complicate the energy integration and its resultant equation. To reiterate, the problem will not change if we can correspond Fourier coefficients in the components of the displacement field, but this requires some proof that every tuple forms a solution. In our previous work we have assumed this is the case based on previous torsional buckling models for linear elastic materials and assumptions made in the problem [24].

Otherwise each Fourier coefficient will be scaled by m, n in separate series and $\sum a_{mn}$, $\sum b_{mn}$, and $\sum c_{mn}$ will be impossible to isolate. In this case if we apply the same displacement assumption to the static equilibrium equations a similar issue appears but not in a form that is useful. Integrating over these Fourier series further complicates our intuitive understanding of what it means when we want to integrate from say 0 to $\frac{L_0}{2m}$ in the energy equation to yield nontrivial equations. In this case each term of the Fourier series is integrated over a different domain and therefore the sums of the integrals cannot be equivalent to the integral of the series, a property we would use in order to integrate and remove the eigenvalues m, n. Instead we will need to integrate over a constant domain. With this in mind, we will proceed by integrating the series over a constant domain so that we can use the absolute convergence of the Fourier series to evaluate the integration. We will also separately include the integrations over varying domains using the three-way paired Fourier coefficients in case it is proved to be a valid assumption. There is a complication due to the presence of residual terms from this type of integration such as $b_{00}\frac{\pi}{n}$ which is undefined, so we will take b_{00}, c_{00} to be 0.

6.3 Integrated Energy Terms

6.3.1 Pre-buckling Strain Energy

Integrating over a constant domain without assumption of single term solutions

$$2\int_0^{\frac{\pi}{2}}\int_0^{L_0}\int_{\rho_i}^{\rho_o}\frac{1}{2}C_0(e^{C_7}-1)\,R\,dR\,dZ\,d\Theta=\frac{1}{2}(\rho_o^2-\rho_i^2)\frac{\pi L_0}{2}C_0(e^{C_7}-1) \tag{6.9}$$

If we can assume single term solutions then we have

$$2\int_0^{\frac{\pi}{n}}\int_0^{\frac{L_0}{2m}}\int_{\rho_i}^{\rho_o}\frac{1}{2}C_0(e^{C_7}-1)\,R\,dR\,dZ\,d\Theta=\frac{1}{2}(\rho_o^2-\rho_i^2)\frac{\pi L_0}{2mn}C_0(e^{C_7}-1) \tag{6.10}$$

6.3.2 Buckled Strain Energy

Integrating over a constant domain without assumption of single term solutions

$$2\int_0^{\frac{\pi}{2}}\int_0^{L_0}\int_{\rho_i+\Omega}^{\rho_o+\Omega}\frac{1}{2}C_0[e^{C_1}(1+C_2\Omega+C_3\frac{\partial\Phi}{\partial\Theta}+C_4\frac{\partial\Phi}{\partial Z}+C_5\frac{\partial\eta}{\partial\Theta}+C_6\frac{\partial\eta}{\partial Z})-1]\,R\,dR\,dZ\,d\Theta$$

$$=\frac{1}{2}(\rho_o^2-\rho_i^2)C_0e^{C_1}\frac{\pi L_0}{2}(1-e^{-C_1})+\sum_{n,m=1}^{\infty}\frac{1}{2}(\rho_o^2-\rho_i^2)C_0e^{C_1}(-\frac{C_2L_0}{\pi}a_{mn}\frac{1}{mn}+\frac{C_3L_0}{\pi}b_{mn}\frac{1}{m}$$
$$+C_4b_{mn}\frac{1}{n}+\frac{C_5L_0}{\pi}c_{mn}\frac{1}{m}+C_6c_{mn}\frac{1}{n})[\sin(m\pi+\frac{n\pi}{2})-\sin(\frac{n\pi}{2})] \quad (6.11)$$

If we can assume single term solutions then we have

$$2\int_0^{\frac{\pi}{n}}\int_0^{\frac{L_0}{2m}}\int_{\rho_i+\Omega}^{\rho_o+\Omega}\frac{1}{2}C_0[e^{C_1}(1+C_2\Omega+C_3\frac{\partial\Phi}{\partial\Theta}+C_4\frac{\partial\Phi}{\partial Z}+C_5\frac{\partial\eta}{\partial\Theta}+C_6\frac{\partial\eta}{\partial Z})-1]\,R\,dR\,dZ\,d\Theta$$

$$=\frac{1}{2}(\rho_o^2-\rho_i^2)C_0e^{C_1}\frac{\pi L_0}{2mn}(1-e^{-C_1})+\frac{1}{2}(\rho_o^2-\rho_i^2)C_0e^{C_1}(-\frac{C_2L_0}{\pi}a_{mn}\frac{1}{mn}+\frac{C_3L_0}{\pi}b_{mn}\frac{1}{m}$$
$$+C_4b_{mn}\frac{1}{n}+\frac{C_5L_0}{\pi}c_{mn}\frac{1}{m}+C_6c_{mn}\frac{1}{n})[\sin(\frac{\pi}{2}+\pi)-\sin(\frac{\pi}{2})] \quad (6.12)$$

6.3.3 External Work

Tension Work

$$2N\int_0^{\frac{\pi}{2}}\int_{\rho_i}^{\rho_o}\eta\,R\,dR\,d\Theta|_0^{L_0}=(\rho_o^2-\rho_i^2)N\sum_{n,m=1}^{\infty}c_{mn}\frac{1}{n}[\sin(m\pi+\frac{n\pi}{2})-\sin(\frac{n\pi}{2})] \quad (6.13)$$

$$2N\int_0^{\frac{\pi}{n}}\int_{\rho_i}^{\rho_o}\eta\,R\,dR\,d\Theta|_0^{\frac{L_0}{2m}}=(\rho_o^2-\rho_i^2)Nc_{mn}\frac{1}{n}[\sin(\frac{\pi}{2}+\pi)-\sin(\frac{\pi}{2})] \quad (6.14)$$

Torsion Work

$$2T\int_0^{\frac{\pi}{2}}\int_{\rho_i}^{\rho_o}\Phi\, R\, dR\, d\Theta|_0^{L_0}= (\rho_o^2-\rho_i^2)T\sum_{n,m=1}^{\infty} b_{mn}\frac{1}{n}[\sin(m\pi+\frac{n\pi}{2})-\sin(\frac{n\pi}{2})] \tag{6.15}$$

$$2T\int_0^{\frac{\pi}{n}}\int_{\rho_i}^{\rho_o}\Phi\, R\, dR\, d\Theta|_0^{\frac{L_0}{2m}}= (\rho_o^2-\rho_i^2)Tb_{mn}\frac{1}{n}[\sin(\frac{\pi}{2}+\pi)-\sin(\frac{\pi}{2})] \tag{6.16}$$

Pressure Work

$$2p\rho_i\int_0^{\frac{\pi}{2}}\int_0^{L_0}\Omega\, dZ\, d\Theta = 2p\rho_i\frac{L_0}{\pi}\sum_{n,m=1}^{\infty} a_{mn}\frac{1}{mn}[\sin(m\pi+\frac{n\pi}{2})-\sin(\frac{n\pi}{2})] \tag{6.17}$$

$$2p\rho_i\int_0^{\frac{\pi}{n}}\int_0^{\frac{L_0}{2m}}\Omega\, dZ\, d\Theta = 2p\rho_i\frac{L_0}{\pi}a_{mn}\frac{1}{mn}[\sin(\frac{\pi}{2}+\pi)-\sin(\frac{\pi}{2})] \tag{6.18}$$

6.3.4 Approximation of Extremum

Inserting (6.9, 6.11, 6.13, 6.15, 6.17) into (6.7)

$$\begin{aligned}0 = {} & \frac{1}{2}(\rho_o^2-\rho_i^2)C_0\frac{\pi L_0}{2}[(e^{C_1}-1)-(e^{C_7}-1)] + \sum_{n,m=1}^{\infty}[\frac{1}{2}(\rho_o^2-\rho_i^2)C_0e^{C_1}(\frac{C_2L_0}{\pi}a_{mn}\frac{1}{mn}+\frac{C_3L_0}{\pi}b_{mn}\frac{1}{m}\\ & + C_4b_{mn}\frac{1}{n}+\frac{C_5L_0}{\pi}c_{mn}\frac{1}{m}+C_6c_{mn}\frac{1}{n})-(\rho_o^2-\rho_i^2)Nc_{mn}\frac{1}{n}\\ & -(\rho_o^2-\rho_i^2)Tb_{mn}\frac{1}{n}-2p\rho_i\frac{L_0}{\pi}a_{mn}\frac{1}{mn}][\sin(m\pi+\frac{n\pi}{2})-\sin(\frac{n\pi}{2})]\end{aligned} \tag{6.19}$$

Inserting (6.10, 6.12, 6.14, 6.16, 6.18) into (6.7)

$$
\begin{aligned}
0 = & \frac{1}{2}(\rho_o^2 - \rho_i^2)C_0\frac{\pi L_0}{2}[(e^{C_1} - 1) - (e^{C_7} - 1)] + [\frac{1}{2}(\rho_o^2 - \rho_i^2)C_0 e^{C_1}(-\frac{C_2 L_0}{\pi}a_{mn}\frac{1}{mn} + \frac{C_3 L_0}{\pi}b_{mn}\frac{1}{m} \\
& + C_4 b_{mn}\frac{1}{n} + \frac{C_5 L_0}{\pi}c_{mn}\frac{1}{m} + C_6 c_{mn}\frac{1}{n}) - (\rho_o^2 - \rho_i^2)N c_{mn}\frac{1}{n} \\
& - (\rho_o^2 - \rho_i^2)T b_{mn}\frac{1}{n} - 2p\rho_i\frac{L_0}{\pi}a_{mn}\frac{1}{mn}][\sin(\frac{\pi}{2} + \pi) - \sin(\frac{\pi}{2})]
\end{aligned}
\tag{6.20}
$$

After cancelling terms and rearranging (6.20) becomes

$$
\begin{aligned}
0 = & \frac{1}{2}[-C_2(1 - e^{-C_1})a_{mn} + (C_3 n + C_4\frac{m\pi}{L_0})b_{mn} + (C_5 n + C_6\frac{m\pi}{L_0})c_{mn}]C_0 e^{C_1} \\
& - (2p\frac{\rho_i}{(\rho_o^2 - \rho_i^2)}a_{mn} + T\frac{m\pi}{L_0}b_{mn} + N\frac{m\pi}{L_0}c_{mn})
\end{aligned}
\tag{6.21}
$$

Remark. It is not immediately obvious that $C_1 = C_7$ with these particular limits of integration because the trigonometric terms evaluate to a constant (-2 by construction) and therefore we might expect $\frac{1}{2}(\rho_o^2 - \rho_i^2)\frac{\pi L_0}{2mn}C_0(e^{C_1} - e^{C_7})$ to be nonzero. However, integrating over other parts of the material volume will yield trigonometric functions that depend on the eigenvalues m, n thereby yielding two separate equations derived from the energy conservation (one over the sinusoidal terms and one over the non-trigonometric terms such that C_1 is necessarily C_7). Intuitively this makes sense because the buckled state is defined as the pre-buckled state plus a perturbation and therefore the constant C_7 from the pre-buckling state must appear in the buckled strain energy density function. This can be further validated by evaluating C_7 directly.

Chapter 7: EQUATIONS OF EQUILIBRIUM FROM ELASTICITY THEORY

This chapter is presented to provide context to the solution methods of Chapter 8. In it we derive the equilibrium equations from elasticity theory using a shell assumption. The static equilibrium (with no acting body forces) can be conveniently written in terms of the nominal stress **S** as $\nabla \cdot \mathbf{S} = 0$. Using the first and second Piola-Kirchoff stress tensors **P** and **Q** respectively and the Cauchy stress $\boldsymbol{\sigma}$ we can rewrite **S** as follows

$$\mathbf{S} = \mathbf{P}^T = (\mathbf{FQ})^T = \mathbf{Q}^T\mathbf{F}^T = \mathbf{F}^{-1}\boldsymbol{\sigma}^T\mathbf{F}^{-T}\mathbf{F}^T = \mathbf{F}^{-1}\mathbf{F}\frac{\partial\Psi}{\partial\mathbf{E}}\mathbf{F}^T = \frac{\partial\Psi}{\partial\mathbf{E}}\mathbf{F}^T \tag{7.1}$$

That is

$$\mathbf{S} = \begin{bmatrix} \frac{\partial\Psi}{\partial E_{RR}}F_{RR} + \frac{\partial\Psi}{\partial E_{R\Theta}}F_{R\Theta} + \frac{\partial\Psi}{\partial E_{RZ}}F_{RZ} & \frac{\partial\Psi}{\partial E_{R\Theta}}F_{\Theta\Theta} + \frac{\partial\Psi}{\partial E_{RZ}}F_{\Theta Z} & \frac{\partial\Psi}{\partial E_{R\Theta}}F_{Z\Theta} + \frac{\partial\Psi}{\partial E_{RZ}}F_{ZZ} \\ \frac{\partial\Psi}{\partial E_{\Theta R}}F_{RR} + \frac{\partial\Psi}{\partial E_{\Theta\Theta}}F_{R\Theta} + \frac{\partial\Psi}{\partial E_{\Theta Z}}F_{RZ} & \frac{\partial\Psi}{\partial E_{\Theta\Theta}}F_{\Theta\Theta} + \frac{\partial\Psi}{\partial E_{\Theta Z}}F_{\Theta Z} & \frac{\partial\Psi}{\partial E_{\Theta\Theta}}F_{Z\Theta} + \frac{\partial\Psi}{\partial E_{\Theta Z}}F_{ZZ} \\ \frac{\partial\Psi}{\partial E_{ZR}}F_{RR} + \frac{\partial\Psi}{\partial E_{Z\Theta}}F_{R\Theta} + \frac{\partial\Psi}{\partial E_{ZZ}}F_{RZ} & \frac{\partial\Psi}{\partial E_{Z\Theta}}F_{\Theta\Theta} + \frac{\partial\Psi}{\partial E_{ZZ}}F_{\Theta Z} & \frac{\partial\Psi}{\partial E_{Z\Theta}}F_{Z\Theta} + \frac{\partial\Psi}{\partial E_{ZZ}}F_{ZZ} \end{bmatrix} \tag{7.2}$$

Note that we can add the hydrostatic pressure term H so that $\hat{S}_{ij} = S_{ij} + H\delta_{ij}$. Taking $\nabla \cdot \mathbf{S} = 0$ in cylindrical coordinates we have

$$\begin{aligned} 0 &= \frac{\partial S_{RR}}{\partial R} + \frac{1}{R}\frac{\partial S_{\Theta R}}{\partial\Theta} + \frac{\partial S_{ZR}}{\partial Z} + \frac{1}{R}(S_{RR} - S_{\Theta\Theta}) \\ 0 &= \frac{\partial S_{R\Theta}}{\partial R} + \frac{1}{R}\frac{\partial S_{\Theta\Theta}}{\partial\Theta} + \frac{\partial S_{Z\Theta}}{\partial Z} + \frac{1}{R}(S_{R\Theta} + S_{\Theta R}) \\ 0 &= \frac{\partial S_{RZ}}{\partial R} + \frac{1}{R}\frac{\partial S_{\Theta Z}}{\partial\Theta} + \frac{\partial S_{ZZ}}{\partial Z} + \frac{1}{R}S_{RZ} \end{aligned} \tag{7.3}$$

By assuming the buckled configuration mapping in Chapter 4 we derive the internal stresses above as functions of our perturbations such that $S_{ij} = S_{ij}(\Theta, Z)$.

We wish to apply boundary conditions (the applied loads) to the above equilibrium equations to establish a relation between the critical angle of twist, loading parameters, geometric parameters, and material parameters. However, this is a difficult task. The thin-wall assumption and the way it has been treated above requires discrete loading conditions since the stress distribution across the thickness of a thin-walled vessel is constant. The internal pressure would necessarily be redistributed longitudinally as it is in traditional thin-wall formulations. However, since we will include a small wall thickness and a small hydrostatic pressure distribution, we require either the entire pressure to be distributed through the wall thickness, an external pressure applied to the exterior of the vessel to balance the rest of the pressure that was not distributed, or the remaining pressure that was not distributed through the wall to be distributed longitudinally.

The boundary conditions at the interior and exterior lateral faces should yield similar resultant equations, otherwise we require trivialities at the interior face that are not useful. To prevent trivialities we would require the inclusion of a phantom exterior pressure, and indeed this is a problem common to all thin-walled vessels since the loading is necessarily discrete (interior pressure is balanced by longitudinal forces). An alternative solution to this is to simply define the way in which we wish to model the hydrostatic pressure and focus on the boundary conditions at the interior of the vessel. In addition to the complications arising from the hydrostatic pressure, the use of the nominal stress greatly complicates application of the external traction requiring the use of Nanson's relations.

7.1 Stress Measures

Recall the buckled deformation Jacobian matrix and strain components

$$\mathbf{F} = \begin{bmatrix} \frac{R_0\Theta_0}{\rho_0\pi\lambda\Lambda} & \frac{1}{R_0}\frac{\partial\Omega}{\partial\Theta} & \frac{\partial\Omega}{\partial Z} \\ 0 & \frac{\rho_0+\Omega}{R_0}(\frac{\partial\Phi}{\partial\Theta}+\frac{\pi}{\Theta_0}) & (\rho_0+\Omega)[\frac{\partial\Phi}{\partial Z}+\frac{\gamma_0}{L_0}] \\ 0 & \frac{1}{R_0}\frac{\partial\eta}{\partial\Theta} & \frac{\partial\eta}{\partial Z}+\lambda\Lambda \end{bmatrix} \tag{7.4}$$

$$\mathbf{E} = \frac{1}{2}(\mathbf{F}^T\mathbf{F}-\mathbf{I}) \tag{7.5}$$

$$\begin{aligned} 2E^b_{RR} &= (\frac{R_0\Theta_0}{\rho_0\pi\lambda\Lambda})^2 - 1 \\ 2E^b_{R\Theta} &= \frac{\Theta_0}{\rho_0\pi\lambda\Lambda}\frac{\partial\Omega}{\partial\Theta} \\ 2E^b_{RZ} &= \frac{R_0\Theta_0}{\rho_0\pi\lambda\Lambda}\frac{\partial\Omega}{\partial Z} \\ 2E^b_{\Theta\Theta} &= \frac{\rho_0^2\pi}{R_0^2\Theta_0}(2\frac{\partial\Phi}{\partial\Theta}+\frac{\pi}{\Theta_0}) + 2\frac{\rho_0\pi^2}{R_0^2\Theta_0^2}\Omega - 1 \\ 2E^b_{\Theta Z} &= \frac{\rho_0^2}{R_0}(\frac{\gamma_0}{L_0}\frac{\partial\Phi}{\partial\Theta}+\frac{\pi}{\Theta_0}\frac{\partial\Phi}{\partial Z}+\frac{\pi\gamma_0}{\Theta_0 L_0}) + \frac{2\rho_0\pi\gamma_0}{R_0\Theta_0 L_0}\Omega + \frac{\lambda\Lambda}{R_0}\frac{\partial\eta}{\partial\Theta} \\ 2E^b_{ZZ} &= (\rho_0^2+2\Omega\rho_0)\frac{\gamma_0^2}{L_0^2} + 2\frac{\rho_0^2\gamma_0}{L_0}\frac{\partial\Phi}{\partial Z} + 2\lambda\Lambda\frac{\partial\eta}{\partial Z} + \lambda^2\Lambda^2 - 1 \end{aligned} \tag{7.6}$$

We substitute the above components into the Fung strain energy density function to obtain the strain energy density of the buckled vessel Ψ_b.

$$\Psi_b = \frac{1}{2}C(e^Q - 1) \tag{7.7}$$

$$\begin{aligned} Q_b = {} & b_1E_{RR}^2 + b_2E_{\Theta\Theta}^2 + b_3E_{ZZ}^2 + 2b_4E_{RR}E_{\Theta\Theta} + 2b_5E_{\Theta\Theta}E_{ZZ} + 2b_6E_{ZZ}E_{RR} + b_7(E_{R\Theta}^2 + E_{\Theta R}^2) \\ & b_8(E_{\Theta Z}^2 + E_{Z\Theta}^2) + b_9(E_{RZ}^2 + E_{ZR}^2) \end{aligned}$$

By taking the derivative of Ψ_b with respect to the strain components we can derive the nominal and Cauchy stresses of the buckled vessel.

To prevent redundant and convoluted equations we will construct a variable naming system. Let the constant terms from the strain components be denoted K_{ij} and their counterpart functions be denoted $f_{ij} = f_{ij}(\Theta, Z)$ so that $2E_{ij}(\Theta, Z) = K_{ij} + f_{ij}(\Theta, Z)$

$$
\begin{aligned}
K_{RR} &= \frac{R_0^2\Theta_0^2}{\rho_0^2\pi^2\lambda^2\Lambda^2} - 1 & f_{RR} &= 0 \\
K_{R\Theta} &= 0 & f_{R\Theta} &= \frac{\Theta_0}{\rho_0\pi\lambda\Lambda}\Omega^\Theta \\
K_{RZ} &= 0 & f_{RZ} &= \frac{R_0\Theta_0}{\rho_0\pi\lambda\Lambda}\Omega^Z \\
K_{\Theta\Theta} &= \frac{\rho_0^2\pi^2}{R_0^2\Theta_0^2} - 1 & f_{\Theta\Theta} &= \frac{\rho_0^2\pi}{R_0^2\Theta_0}2\Phi^\Theta + 2\frac{\rho_0\pi^2}{R_0^2\Theta_0^2}\Omega \\
K_{\Theta Z} &= \frac{\rho_0^2\pi\gamma_0}{R_0\Theta_0L_0} & f_{\Theta Z} &= \frac{\rho_0^2}{R_0}(\frac{\gamma_0}{L_0}\Phi^\Theta + \frac{\pi}{\Theta_0}\Phi^Z) + \frac{2\rho_0\pi\gamma_0}{R_0\Theta_0L_0}\Omega + \frac{\lambda\Lambda}{R_0}\eta^\Theta \\
K_{ZZ} &= \frac{\rho_0^2\gamma_0^2}{L_0^2} + \lambda^2\Lambda^2 - 1 & f_{ZZ} &= 2\rho_0\frac{\gamma_0^2}{L_0^2}\Omega + 2\frac{\rho_0^2\gamma_0}{L_0}\Phi^Z + 2\lambda\Lambda\eta^Z
\end{aligned}
\tag{7.8}
$$

We will use a rank seven coefficient tensor $D^{p^{lm}}_{n^k_{ij}}$ to repurpose partial derivative coefficients that appear in the nominal stress components. The subscript n^k_{ij} denotes the source from which the coefficient is derived and the superscript p^{lm} denotes the partial derivatives to which the coefficient correspond. The terms $\frac{\partial e^Q}{\partial Z}$, $\frac{\partial e^Q}{\partial \Theta}$ show up repeatedly in the partial derivatives of the components S_{ij} so they are left in symbolic representation within each nominal stress component and evaluated alongside them at the end. However, their coefficients are still distinguished within each nominal stress component's derivative using the coefficient tensor. The K_{ij}, f_{ij} terms above serve to elucidate the derivation of each coefficient tensor component since they will be rewritten as part of the larger coefficient abstraction system. The variations are

$$
\begin{aligned}
& n \in (\Omega, \Phi, \eta, e, K) && i, j \in (1, 2, 3) \sim (R, \Theta, Z) \\
& p \in (S, E, F, e) && k, l, m \in (2, 3) \sim (\Theta, Z)
\end{aligned}
\tag{7.9}
$$

For example, the coefficient of $\Phi^{\Theta\Theta}$ present in the nominal stress partial $\frac{\partial S_{\Theta Z}}{\partial Z}$ will be $D^{\Phi^{22}}_{S^3_{23}}$. The coefficient of $\frac{\partial e^Q}{\partial Z}$ within $\frac{\partial S_{\Theta Z}}{\partial Z}$ will be $D^{e^3}_{S^3_{23}}$ which will be multiplied by $D^{p^{lm}}_{e^3}$ in the resultant equilibrium equations. Indices not needed (for example if there is no derivative or the derivative is first order) will be left blank. Note that E above refers to the strain components E_{ij}, F refers to the deformation gradient components F_{ij}, and K implies the term is a standalone constant.

7.1.1 Nominal Stresses

$$\frac{\partial S_{\Theta R}}{\partial \Theta}\frac{2}{C_0}e^{-Q} = D^{\Omega^{22}}_{S^2_{21}}\Omega^{\Theta\Theta} + D^{\Omega^{32}}_{S^2_{21}}\Omega^{Z\Theta} \tag{7.10}$$

$$\frac{\partial S_{ZR}}{\partial Z}\frac{2}{C_0}e^{-Q} = D^{\Omega^{23}}_{S^3_{31}}\Omega^{\Theta Z} + D^{\Omega^{33}}_{S^3_{31}}\Omega^{ZZ} \tag{7.11}$$

$$S_{RR}\frac{2}{C_0}e^{-Q} = D^{K}_{S_{11}} + D^{\Omega}_{S_{11}}\Omega + D^{\Phi^2}_{S_{11}}\Phi^{\Theta} + D^{\Phi^3}_{S_{11}}\Phi^{Z} + D^{\eta^3}_{S_{11}}\eta^{Z} \tag{7.12}$$

$$S_{\Theta\Theta}\frac{2}{C_0}e^{-Q} = D^{K}_{S_{22}} + D^{\Omega}_{S_{22}}\Omega + D^{\Phi^2}_{S_{22}}\Phi^{\Theta} + D^{\Phi^3}_{S_{22}}\Phi^{Z} + D^{\eta^2}_{S_{22}}\eta^{\Theta} + D^{\eta^3}_{S_{22}}\eta^{Z} \tag{7.13}$$

$$\frac{\partial S_{\Theta\Theta}}{\partial \Theta}\frac{2}{C_0}e^{-Q} = D^{\Omega^2}_{S^2_{22}}\Omega^{\Theta} + D^{\Phi^{22}}_{S^2_{22}}\Phi^{\Theta\Theta} + D^{\Phi^{32}}_{S^2_{22}}\Phi^{Z\Theta} + D^{\eta^{22}}_{S_{22}2}\eta^{\Theta\Theta} + D^{\eta^{32}}_{S^2_{22}}\eta^{Z\Theta} + D^{e^2}_{S^2_{22}}\frac{\partial e^Q}{\partial \Theta}e^{-Q} \tag{7.14}$$

$$\frac{\partial S_{Z\Theta}}{\partial Z}\frac{2}{C_0}e^{-Q} = D^{\Omega^3}_{S^3_{32}}\Omega^{Z} + D^{\Phi^{23}}_{S^3_{32}}\Phi^{\Theta Z} + D^{\Phi^{33}}_{S^3_{32}}\Phi^{ZZ} + D^{\eta^{33}}_{S^{23}_{32}}\eta^{\Theta Z} + D^{\eta^{33}}_{S^3_{32}}\eta^{ZZ} + D^{e^3}_{S^3_{32}}\frac{\partial e^Q}{\partial Z}e^{-Q} \tag{7.15}$$

$$S_{R\Theta}\frac{2}{C_0}e^{-Q} = 0 \tag{7.16}$$

$$S_{\Theta R}\frac{2}{C_0}e^{-Q} = D^{\Omega^2}_{S_{21}}\Omega^{\Theta} + D^{\Omega^3}_{S_{21}}\Omega^{Z} \tag{7.17}$$

$$\frac{\partial S_{\Theta Z}}{\partial \Theta}\frac{2}{C_0}e^{-Q} = D^{\Omega^2}_{S^2_{23}}\Omega^{\Theta} + D^{\Phi^{22}}_{S^2_{23}}\Phi^{\Theta\Theta} + D^{\Phi^{32}}_{S^2_{23}}\Phi^{Z\Theta} + D^{\eta^{22}}_{S^2_{23}}\eta^{\Theta\Theta} + D^{\eta^{32}}_{S^2_{23}}\eta^{Z\Theta} + D^{e^2}_{S^2_{23}}\frac{\partial e^{Q}}{\partial \Theta}e^{-Q} \tag{7.18}$$

$$\frac{\partial S_{ZZ}}{\partial Z}\frac{2}{C_0}e^{-Q} = D^{\Omega^3}_{S^3_{33}}\Omega^{Z} + D^{\Phi^{23}}_{S^3_{33}}\Phi^{\Theta Z} + D^{\Phi^{33}}_{S^3_{33}}\Phi^{ZZ} + D^{\eta^{23}}_{S^3_{33}} + D^{\eta^{33}}_{S^3_{33}}\eta^{ZZ} + D^{e^3}_{S^2_{33}}\frac{\partial e^{Q}}{\partial Z}e^{-Q} \tag{7.19}$$

$$S_{RZ}\frac{2}{C_0}e^{-Q} = 0 \tag{7.20}$$

$$\frac{\partial e^{Q}}{\partial \Theta}\frac{2}{C_0}e^{-Q} = D^{\Omega^2}_{e^2}\Omega^{\Theta} + D^{\Phi^{22}}_{e^2}\Phi^{\Theta\Theta} + D^{\Phi^{32}}_{e^2}\Phi^{Z\Theta} + D^{\eta^{22}}_{e^2}\eta^{\Theta\Theta} + D^{\eta^{32}}_{e^2}\eta^{Z\Theta} \tag{7.21}$$

$$\frac{\partial e^{Q}}{\partial Z}\frac{2}{C_0}e^{-Q} = D^{\Omega^3}_{e^3}\Omega^{Z} + D^{\Phi^{23}}_{e^3}\Phi^{\Theta Z} + D^{\Phi^{33}}_{e^3}\Phi^{ZZ} + D^{\eta^{23}}_{e^3}\eta^{\Theta Z} + D^{\eta^{33}}_{e^3}\eta^{ZZ} \tag{7.22}$$

7.1.2 Cauchy Stresses

Note that $\sigma_{ij} = F_{ik}S_{kj}$ and $\sigma_{ij} = \sigma_{ji}$

$$\sigma_{RR}\frac{2}{C_0}e^{-Q} = D^K_{F_{11}}(D^K_{S_{11}} + D^{\Omega}_{S_{11}}\Omega + D^{\Phi^2}_{S_{11}}\Phi^{\Theta} + D^{\Phi^3}_{S_{11}}\Phi^Z + D^{\eta^3}_{S_{11}}\eta^Z) \tag{7.23}$$

$$\sigma_{R\Theta}\frac{2}{C_0}e^{-Q} = \frac{1}{R_0}\Omega^{\Theta}(D^K_{S_{22}}) + \Omega^Z(D^K_{S_{23}}) \tag{7.24}$$

$$\sigma_{RZ}\frac{2}{C_0}e^{-Q} = \frac{1}{R_0}\Omega^{\Theta}(D^K_{S_{23}}) + \Omega^Z(D^K_{S_{33}}) \tag{7.25}$$

$$\begin{aligned}\sigma_{\Theta\Theta}\frac{2}{C_0}e^{-Q} &= D^K_{F_{22}}(D^K_{S_{22}} + D^{\Omega}_{S_{22}}\Omega + D^{\Phi^2}_{S_{22}}\Phi^{\Theta} + D^{\Phi^3}_{S_{22}}\Phi^Z + D^{\eta^2}_{S_{22}}\eta^{\Theta} + D^{\eta^3}_{S_{22}}\eta^Z)\\ &\quad + D^K_{F_{23}}(D^K_{S_{32}} + D^{\Omega^3}_{S^3_{32}}\Omega + D^{\Phi^{23}}_{S^3_{32}}\Phi^{\Theta} + D^{\Phi^{33}}_{S^3_{32}}\Phi^Z + D^{\eta^{33}}_{S^{23}_{32}}\eta^{\Theta} + D^{\eta^{33}}_{S^3_{32}}\eta^Z)\\ &\quad + (\frac{\rho_0}{R_0}\Phi^{\Theta} + \frac{\pi}{R_0\Theta_0}\Omega)(D^K_{S_{22}}) + (\rho_0\Phi^Z + \frac{\gamma_0}{L_0}\Omega)(D^K_{S_{32}})\end{aligned} \tag{7.26}$$

$$\begin{aligned}\sigma_{\Theta Z}\frac{2}{C_0}e^{-Q} &= D^K_{F_{22}}(D^K_{S_{23}} + D^{\Omega^2}_{S^2_{23}}\Omega + D^{\Phi^{22}}_{S^2_{23}}\Phi^{\Theta} + D^{\Phi^{32}}_{S^2_{23}}\Phi^Z + D^{\eta^{22}}_{S^2_{23}}\eta^{\Theta} + D^{\eta^{32}}_{S^2_{23}}\eta^Z)\\ &\quad + D^K_{F_{23}}(D^K_{S_{33}} + D^{\Omega^3}_{S^3_{33}}\Omega + D^{\Phi^{23}}_{S^3_{33}}\Phi^{\Theta} + D^{\Phi^{33}}_{S^3_{33}}\Phi^Z + D^{\eta^2}_{S^3_{33}} + D^{\eta^{33}}_{S^3_{33}}\eta^Z)\\ &\quad + (\frac{\rho_0}{R_0}\Phi^{\Theta} + \frac{\pi}{R_0\Theta_0}\Omega)(D^K_{S_{23}}) + (\rho_0\Phi^Z + \frac{\gamma_0}{L_0}\Omega)(D^K_{S_{33}})\end{aligned} \tag{7.27}$$

$$\begin{aligned}\sigma_{ZZ}\frac{2}{C_0}e^{-Q} &= D^K_{F_{33}}(D^K_{S_{33}} + D^{\Omega^3}_{S^3_{33}}\Omega + D^{\Phi^{23}}_{S^3_{33}}\Phi^{\Theta} + D^{\Phi^{33}}_{S^3_{33}}\Phi^Z + D^{\eta^2}_{S^3_{33}} + D^{\eta^{33}}_{S^3_{33}}\eta^Z)\\ &\quad + \frac{1}{R_0}\eta^{\Theta}(D^K_{S_{23}}) + \eta^Z(D^K_{S_{33}})\end{aligned} \tag{7.28}$$

7.2 Hybrid Thin/Thick-Wall Hydrostatic Pressure Distribution

In traditional thin-wall inflation problems the hydrostatic pressure is applied discretely so that the distribution is zero across the wall thickness but is fully distributed longitudinally as a constant.

Forcing the approximation $D^K_{S_{11}} \approx D^K_{S_{22}}$ also forces a constant pressure distribution within the wall. We could therefore apply a pure thin-wall condition by allowing the above approximation, but the terms do not cancel out nicely unless we violate some assumptions about the independence of material and geometric parameters.

It therefore stands to reason we should include some sort of a distribution within the wall thereby obtaining a hybrid thick- and thin-wall model but only with respect to the hydrostatic pressure. If we assume the existence of a variable hydrostatic pressure distribution we can introduce a Lagrangian multiplier H to zero the radial equation of motion. The Lagrangian multiplier is exactly the hydrostatic pressure within the vessel at any material point and is subsequently added onto each normal stress component such that $\hat{S}_{ij} = S_{ij} + H\delta_{ij}$.

We must make some assumptions about the distribution of the hydrostatic pressure in our hybrid model. We will assume that the pressure is completely distributed through the vessel wall so that at the interior wall the hydrostatic pressure is equal to the applied internal pressure and at the exterior wall it is equal to zero. This is true in the thin and thick wall cases, with the thin-wall condition requiring a longitudinal distribution of the pressure. We can adapt this to our hybrid model by allowing a small distribution across the wall of a fraction of the interior pressure and the rest distributed longitudinally. The amount that is distributed longitudinally will be $H(\rho_o)$. To apply the boundary conditions note that S_{RR} is independent of the radius and hence the transmural pressure $p = S_{RR} + H(\rho_o) - S_{RR} - H(\rho_i) = H(\rho_o) - H(\rho_i)$.

7.2.1 Buckled Hydrostatic Equilibrium

From the buckling equations we obtain the following equation in the radial direction (using coordinate system (R, Θ, Z) and the nominal instead of Cauchy stress)

$$\begin{aligned}0 = R\frac{\partial H}{\partial R}\frac{2}{C_0}e^{-Q_b} + (D^K_{S_{11}} - D^K_{S_{22}}) + (D^\Omega_{S_{11}} - D^\Omega_{S_{22}})\Omega + D^{\Omega^{22}}_{S^2_{21}}\Omega^{\Theta\Theta} + (D^{\Omega^{32}}_{S^2_{21}} + RD^{\Omega^{23}}_{S^3_{31}})\Omega^{\Theta Z} \\ + RD^{\Omega^{33}}_{S^3_{31}}\Omega^{ZZ} + (D^{\Phi^2}_{S_{11}} - D^{\Phi^2}_{S_{22}})\Phi^\Theta + (D^{\Phi^3}_{S_{11}} - D^{\Phi^3}_{S_{22}})\Phi^Z - D^{\eta^2}_{S_{22}}\eta^\Theta + (D^{\eta^3}_{S_{11}} - D^{\eta^3}_{S_{22}})\eta^Z\end{aligned} \tag{7.29}$$

If we express e^{Q_b} as a power series, recall $e^{Q_b} = e^{C_1}(1 + C_2\Omega + C_3\frac{\partial\Phi}{\partial\Theta} + C_4\frac{\partial\Phi}{\partial Z} + C_5\frac{\partial\eta}{\partial\Theta} + C_6\frac{\partial\eta}{\partial Z})$, we will after reducing higher order terms have two equations remaining from the constant terms and trigonometric terms shown respectively below.

$$\begin{aligned}
0 &= R\frac{\partial H_n}{\partial R}\frac{2}{C_0}e^{-C_1} + (D^K_{S_{11}} - D^K_{S_{22}}) \\
&= R\frac{\partial H_n}{\partial R}2 + [b_1(\frac{R_0^2\Theta_0^2}{R^2\pi^2\lambda^2\Lambda^2} - 1) + b_4(\frac{R^2\pi^2}{R_0^2\Theta_0^2} - 1) + b_6(\frac{R^2\gamma_0^2}{L_0^2} + \lambda^2\Lambda^2 - 1)]\frac{R_0\Theta_0}{R\pi\lambda\Lambda}C_0e^{C_1} \\
&\quad - [b_4(\frac{R_0^2\Theta_0^2}{R^2\pi^2\lambda^2\Lambda^2} - 1) + b_2(\frac{R^2\pi^2}{R_0^2\Theta_0^2} - 1) + b_5(\frac{R^2\gamma_0^2}{L_0^2} + \lambda^2\Lambda^2 - 1)]\frac{R\pi}{R_0\Theta_0}C_0e^{C_1} \\
&\quad - b_8(\frac{R^2\pi\gamma_0}{R_0\Theta_0L_0})\frac{R\gamma_0}{L_0}C_0e^{C_1} \qquad (7.30)
\end{aligned}$$

$$\begin{aligned}
0 &= R\frac{\partial H_t}{\partial R}\frac{2}{C_0}e^{-C_1} + (D^K_{S_{11}} - D^K_{S_{22}})(C_2\Omega + C_3\Phi^\Theta + C_4\Phi^Z + C_5\eta^\Theta + C_6\eta^Z) + (D^\Omega_{S_{11}} - D^\Omega_{S_{22}})\Omega \\
&\quad + D^{\Omega^{22}}_{S^2_{21}}\Omega^{\Theta\Theta} + (D^{\Omega^{32}}_{S^2_{21}} + RD^{\Omega^{23}}_{S^3_{31}})\Omega^{\Theta Z} + RD^{\Omega^{33}}_{S^3_{31}}\Omega^{ZZ} + (D^{\Phi^2}_{S_{11}} - D^{\Phi^2}_{S_{22}})\Phi^\Theta + (D^{\Phi^3}_{S_{11}} - D^{\Phi^3}_{S_{22}})\Phi^Z \\
&\quad - D^{\eta^2}_{S_{22}}\eta^\Theta + (D^{\eta^3}_{S_{11}} - D^{\eta^3}_{S_{22}})\eta^Z \qquad (7.31)
\end{aligned}$$

We must now carefully make assumptions for the general form of H. If we assume H is completely trigonometric we have a contradiction unless we make the approximation that $D^K_{S_{11}} \approx D^K_{S_{22}}$; we will call this result Case 1. Recall we obtain Case 1 if we enforce the full thin-wall condition or if we desire a constant hydrostatic pressure distribution. If we assume H is not trigonometric then we require all trigonometric terms from the second equation to cancel; we will call this Case 2. We will prove without any prior assumption of the form of H that the assumptions for Case 2 are valid for the hybrid model such that

$$\begin{aligned}
0 &= (D^K_{S_{11}} - D^K_{S_{22}})(C_2\Omega + C_3\Phi^\Theta + C_4\Phi^Z + C_5\eta^\Theta + C_6\eta^Z) + (D^\Omega_{S_{11}} - D^\Omega_{S_{22}})\Omega + D^{\Omega^{22}}_{S^2_{21}}\Omega^{\Theta\Theta} \\
&\quad + (D^{\Omega^{32}}_{S^2_{21}} + \rho_0 D^{\Omega^{23}}_{S^3_{31}})\Omega^{\Theta Z} + \rho_0 D^{\Omega^{33}}_{S^3_{31}}\Omega^{ZZ} + (D^{\Phi^2}_{S_{11}} - D^{\Phi^2}_{S_{22}})\Phi^\Theta + (D^{\Phi^3}_{S_{11}} - D^{\Phi^3}_{S_{22}})\Phi^Z - D^{\eta^2}_{S_{22}}\eta^\Theta \\
&\quad + (D^{\eta^3}_{S_{11}} - D^{\eta^3}_{S_{22}})\eta^Z \qquad (7.32)
\end{aligned}$$

Proof. Suppose H has trigonometric and non-trigonometric parts such that $H = H_t + H_n$. Then

$H = H(R, \Theta, Z)$ and

$$0 = R\frac{\partial H_t}{\partial R}\frac{2}{C_0}e^{-C_1} + (D^K_{S_{11}} - D^K_{S_{22}})(C_2\Omega + C_3\Phi^\Theta + C_4\Phi^Z + C_5\eta^\Theta + C_6\eta^Z) + (D^\Omega_{S_{11}} - D^\Omega_{S_{22}})\Omega$$
$$+ D^{\Omega^{22}}_{S^2_{21}}\Omega^{\Theta\Theta} + (D^{\Omega^{32}}_{S^2_{21}} + RD^{\Omega^{23}}_{S^3_{31}})\Omega^{\Theta Z} + RD^{\Omega^{33}}_{S^3_{31}}\Omega^{ZZ} + (D^{\Phi^2}_{S_{11}} - D^{\Phi^2}_{S_{22}})\Phi^\Theta + (D^{\Phi^3}_{S_{11}} - D^{\Phi^3}_{S_{22}})\Phi^Z$$
$$- D^{\eta^2}_{S_{22}}\eta^\Theta + (D^{\eta^3}_{S_{11}} - D^{\eta^3}_{S_{22}})\eta^Z \tag{7.33}$$

$$0 = R\frac{\partial H_n}{\partial R}2 + [b_1(\frac{R_0^2\Theta_0^2}{\rho_0^2\pi^2\lambda^2\Lambda^2} - 1) + b_4(\frac{\rho_0^2\pi^2}{R_0^2\Theta_0^2} - 1) + b_6(\frac{\rho_0^2\gamma_0^2}{L_0^2} + \lambda^2\Lambda^2 - 1)]\frac{R_0\Theta_0}{\rho_0\pi\lambda\Lambda}C_0e^{C_1}$$
$$- [b_4(\frac{R_0^2\Theta_0^2}{\rho_0^2\pi^2\lambda^2\Lambda^2} - 1) + b_2(\frac{\rho_0^2\pi^2}{R_0^2\Theta_0^2} - 1) + b_5(\frac{\rho_0^2\gamma_0^2}{L_0^2} + \lambda^2\Lambda^2 - 1)]\frac{\rho_0\pi}{R_0\Theta_0}C_0e^{C_1}$$
$$- b_8(\frac{\rho_0^2\pi\gamma_0}{R_0\Theta_0L_0})\frac{\rho_0\gamma_0}{L_0}C_0e^{C_1} \tag{7.34}$$

If the hydrostatic pressure distribution is independent of (Θ, Z) then $H = H_n(R)$ trivially. Applying our hybrid model the transmural pressure will be $p_o - p_i = -H(\rho_o) - (-p) = p - H(\rho_o)$. Here we have essentially added an external pressure to balance the pressure distribution and we will later add $H(\rho_o)$ to the axial equilibrium.

If $H(\rho_0)$ does depend on Θ, Z then the first equation of motion cancels automatically and H introduces trigonometric terms from the first equation into the second and third equations of motion ($\frac{\partial H_t}{\partial \Theta}, \frac{\partial H_t}{\partial Z}$ respectively). However, these terms preserve the original structure of the trigonometric terms from the radial equation of motion and since they all share the same factor ln(R) and R they must cancel so that $\frac{\partial H_t}{\partial \Theta} = \frac{\partial H_t}{\partial Z} = 0$. But this implies that $H = H_n(R)$ again resulting in Case 2. □

Remark. If we evaluate the equilibrium equations at ρ_0 prior to our analysis we will actually obtain a factor R instead of ln(R) in front of these terms, and by a similar argument they must cancel. We would instead derive a linear relation for $H(R)$ and therefore we can consider the logarithmic hydrostatic distribution as the cumulative sum of all of these individual radial contributions. The practical result is the same if we use a logarithmic distribution or a linear one, but the former will displace a couple of very small terms with a coefficient of R^2 into the second and third equations of motion. When evaluating the radial equilibrium equation at different radiuses, we simply change the factor $\frac{1}{R}$ in H to whatever radius the equation is evaluated.

Therefore we will assume $H = H(R)$ without any trigonometric terms, implying the pressure is only slightly distributed logarithmically through the vessel wall. If the radial equilibrium equation is evaluated at $R = \rho_0$ we have

$$\begin{aligned} H(R) = & -\ln(R)[b_1(\frac{R_0^2\Theta_0^2}{\rho_0^2\pi^2\lambda^2\Lambda^2} - 1) + b_4(\frac{\rho_0^2\pi^2}{R_0^2\Theta_0^2} - 1) + b_6(\frac{\rho_0^2\gamma_0^2}{L_0^2} + \lambda^2\Lambda^2 - 1)]\frac{R_0\Theta_0}{\rho_0\pi\lambda\Lambda}\frac{C_0}{2}e^{C_1} \\ & + \ln(R)[b_4(\frac{R_0^2\Theta_0^2}{\rho_0^2\pi^2\lambda^2\Lambda^2} - 1) + b_2(\frac{\rho_0^2\pi^2}{R_0^2\Theta_0^2} - 1) + b_5(\frac{\rho_0^2\gamma_0^2}{L_0^2} + \lambda^2\Lambda^2 - 1)]\frac{\rho_0\pi}{R_0\Theta_0}\frac{C_0}{2}e^{C_1} \\ & + \ln(R)b_8(\frac{\rho_0^2\pi\gamma_0}{R_0\Theta_0L_0})\frac{\rho_0\gamma_0}{L_0}\frac{C_0}{2}e^{C_1} + K \end{aligned} \tag{7.35}$$

The constant K will be given by the traction boundary condition for pressure.

7.3 Buckling Equations

Per the proof the three equations of equilibrium we will use are

$$\begin{aligned}0 &= (D^{K}_{S_{11}} - D^{K}_{S_{22}})(C_2\Omega + C_3\Phi^{\Theta} + C_4\Phi^{Z} + C_5\eta^{\Theta} + C_6\eta^{Z}) + (D^{\Omega}_{S_{11}} - D^{\Omega}_{S_{22}})\Omega + D^{\Omega^{22}}_{S^2_{21}}\Omega^{\Theta\Theta} \\ &+ (D^{\Omega^{32}}_{S^2_{21}} + \rho_0 D^{\Omega^{23}}_{S^3_{31}})\Omega^{\Theta Z} + \rho_0 D^{\Omega^{33}}_{S^3_{31}}\Omega^{ZZ} + (D^{\Phi^2}_{S_{11}} - D^{\Phi^2}_{S_{22}})\Phi^{\Theta} + (D^{\Phi^3}_{S_{11}} - D^{\Phi^3}_{S_{22}})\Phi^{Z} - D^{\eta^2}_{S_{22}}\eta^{\Theta} \\ &+ (D^{\eta^3}_{S_{11}} - D^{\eta^3}_{S_{22}})\eta^{Z}\end{aligned} \tag{7.36}$$

$$\begin{aligned}0 &= \frac{1}{\rho_0}(D^{\Omega^2}_{S^2_{22}} + D^{\Omega^2}_{S_{21}} + D^{e^2}_{S^2_{22}}D^{\Omega^2}_{e^2})\Omega^{\Theta} + \frac{1}{\rho_0}(D^{\Omega^3}_{S_{21}} + \rho_0 D^{\Omega^3}_{S^3_{32}} + \rho_0 D^{e^3}_{S^3_{32}}D^{\Omega^3}_{e^3})\Omega^{Z} \\ &+ \frac{1}{\rho_0}(D^{\Phi^{22}}_{S^2_{22}} + D^{e^2}_{S^2_{22}}D^{\Phi^{22}}_{e^2})\Phi^{\Theta\Theta} + \frac{1}{\rho_0}(D^{\Phi^{32}}_{S^2_{22}} + \rho_0 D^{\Phi^{23}}_{S^3_{32}} + D^{e^2}_{S^2_{22}}D^{\Phi^{32}}_{e^2} + \rho_0 D^{e^3}_{S^3_{32}}D^{\Phi^{23}}_{e^3})\Phi^{\Theta Z} \\ &+ (D^{\Phi^{33}}_{S^3_{32}} + D^{e^3}_{S^3_{32}}D^{\Phi^{33}}_{e^3})\Phi^{ZZ} + \frac{1}{\rho_0}(D^{e^2}_{S^2_{22}}D^{\eta^{22}}_{e^2} + D^{\eta^{22}}_{S^2_{22}})\eta^{\Theta\Theta} \\ &+ \frac{1}{\rho_0}(D^{\eta^{32}}_{S^2_{22}} + D^{e^2}_{S^2_{22}}D^{\eta^{32}}_{e^2} + \rho_0 D^{\eta^{23}}_{S^3_{32}} + \rho_0 D^{e^3}_{S^3_{32}}D^{\eta^{32}}_{e^2})\eta^{\Theta Z} + (D^{\eta^{33}}_{S^3_{32}} + D^{e^3}_{S^3_{32}}D^{\eta^{33}}_{e^3})\eta^{ZZ}\end{aligned} \tag{7.37}$$

$$\begin{aligned}0 &= \frac{1}{\rho_0}(D^{\Omega^2}_{S^2_{23}} + D^{e^2}_{S^2_{23}}D^{\Omega^2}_{e^2})\Omega^{\Theta} + \frac{1}{\rho_0}(\rho_0 D^{\Omega^3}_{S^3_{33}} + D^{\Omega^3}_{S_{13}} + \rho_0 D^{e^3}_{S^3_{33}}D^{\Omega^3}_{e^3})\Omega^{Z} \\ &+ \frac{1}{\rho_0}(D^{\Phi^{22}}_{S^2_{23}} + D^{e^2}_{S^2_{23}}D^{\Phi^{22}}_{e^2})\Phi^{\Theta\Theta} + \frac{1}{\rho_0}(D^{\Phi^{32}}_{S^2_{23}} + \rho_0 D^{\Phi^{23}}_{S^3_{33}} + D^{e^2}_{S^2_{23}}D^{\Phi^{32}}_{e^2} + \rho_0 D^{e^3}_{S^3_{33}}D^{\Phi^{23}}_{e^3})\Phi^{\Theta Z} \\ &+ (D^{\Phi^{33}}_{S^3_{33}} + D^{e^3}_{S^3_{33}}D^{\Phi^{33}}_{e^3})\Phi^{ZZ} + \frac{1}{\rho_0}(D^{\eta^{22}}_{S^2_{23}} + D^{e^2}_{S^2_{23}}D^{\eta^{22}}_{e^2})\eta^{\Theta\Theta} \\ &+ \frac{1}{\rho_0}(D^{\eta^{32}}_{S^2_{23}} + D^{e^2}_{S^2_{23}}D^{\eta^{32}}_{e^2} + \rho_0 D^{\eta^{23}}_{S^3_{33}} + \rho_0 D^{e^3}_{S^3_{33}}D^{\eta^{32}}_{e^3})\eta^{\Theta Z} + (D^{\eta^{33}}_{S^3_{33}} + D^{e^3}_{S^3_{33}}D^{\eta^{33}}_{e^3})\eta^{ZZ}\end{aligned} \tag{7.38}$$

7.4 Boundary Conditions

The boundary conditions for the nominal stresses are expressed in terms of the external traction and the undeformed material configuration as $\vec{t} = \hat{N} \cdot \mathbf{S}$ where $\hat{N}$ is the normal direction of the surface in the undeformed state. By contrast the Cauchy stress boundary conditions are $\vec{t} = \hat{n} \cdot \boldsymbol{\sigma}$ where $\hat{n}$ is the normal vector of the deformed surface. The external traction $\vec{t}$ is obtained by applying an external force to the surface and normalizing by the acting deformed surface area. A traction which depends on the surface normal vector, such as pressure, will change direction throughout deformation. The boundary conditions will therefore be partial differential equations that necessarily incorporate elements from the deformation gradient.

7.4.1 Surface Normal Vectors

We must derive the normal vectors of the deformed material surface regardless of stress measure. However, there are many cancellations if we are to use the Cauchy stress traction boundary conditions combined with the assumption that all applied tractions have the direction of the deformed surface normals. In this case all tractions would be acting as positive or negative pressures and normal vector components from the traction will always cancel. However, in applying the axial tension we want the traction to only be applied on the Z normal face and to be zero everywhere else so that the direction in which the vessel is being pulled is controlled. In this case using the Cauchy stress has limited advantages.

Since we are using the nominal stress the normal vector components in the tractions do not cancel and we must therefore utilize either some vector calculus on parametric surfaces or Nanson's relations. We have outlined both results here, although the parametric surface normals differ slightly on the lateral faces wherein they predict two extra components in the deformed normal vector (projecting this vector onto the radial direction results in the same vector derived from Nanson's relations). The reason for this is still unclear.

Parametric Surface

This method constructs the vessel as the union of two kinds of parametric surfaces (four surfaces total) using vector calculus. The normal vectors are obtained using the configuration mappings to construct two parametric surfaces $\vec{r_l}(\Theta, Z)$ and $\vec{r_e}(\rho, \Theta)$ and taking the cross product of the partial derivatives of $\vec{r}$. The direction of the surface normal vectors during buckling of the lateral surfaces and the ends of the vessel are respectively

$$\hat{n_l} = \pm d_1 < \frac{\pi}{\Theta_0}(\frac{\partial \eta}{\partial Z} + \lambda\Lambda) + \frac{\partial \Phi}{\partial \Theta}\lambda\Lambda - \frac{\partial \eta}{\partial \Theta}\frac{\gamma_0}{L_0}, -\frac{\partial \Omega}{\partial \Theta}\lambda\Lambda, \frac{\partial \Omega}{\partial \Theta}\frac{\gamma_0}{L_0} - \frac{\pi}{\Theta_0}\frac{\partial \Omega}{\partial Z} >$$
$$\hat{n_e} = \pm d_2 < 0, -\frac{\partial \eta}{\partial \Theta}, \frac{\pi}{\Theta_0} + \frac{\partial \Phi}{\partial \Theta} > \quad (7.39)$$

where d_1, d_2 are normalization terms.

Nanson's Relations

This method uses the deformation gradient as a map. If $\mathbf{F}$ is the deformation gradient then a differential elemental area $dA^{(0)}$ with normal $\hat{N}$ is mapped into a deformed differential elemental area $dA^{(1)}$ with normal $\hat{n}$ by $\hat{n}dA^{(1)} = J\mathbf{F}^{-T} \cdot dA^{(0)}\hat{N}$ where $J := det(\mathbf{F})$. The direction of the surface normal vectors during buckling of the lateral surfaces and the ends of the vessel are respectively

$$\hat{n_l} = \pm d_3 < \frac{\pi}{\Theta_0}(\frac{\partial \eta}{\partial Z} + \lambda\Lambda) + \frac{\partial \Phi}{\partial \Theta}\lambda\Lambda - \frac{\partial \eta}{\partial \Theta}\frac{\gamma_0}{L_0}, 0, 0 >$$
$$\hat{n_e} = \pm d_4 < 0, -\frac{\partial \eta}{\partial \Theta}, \frac{\pi}{\Theta_0} + \frac{\partial \Phi}{\partial \Theta} > \quad (7.40)$$

where d_3 is the ratio of the undeformed to deformed lateral surface area and d_4 is the ratio of the undeformed to the deformed end surface area. The end surface normal vectors for both methods are identical, hence d_2 is also the ratio of the undeformed to the deformed cross-sectional area. We can solve for d_3 and d_4 by looking at the Cauchy stress boundary conditions and comparing them to the nominal stress boundary conditions.

7.4.2 Nominal Stress Traction Boundary Conditions

We assume that the internal pressure is applied in the opposite direction of the surface normal at the interior lateral surface and that the axial tension is applied in the direction of the surface normal at the ends of the vessel but only ever acts in the axial direction. First we will look at the hydrostatic pressure $H(R)$.

Parametric Surface Normals

Now applying the condition $\vec{t} = \hat{N} \cdot S$, where $\vec{t} = -p\hat{n}_l$ we obtain the following three relations at $R = \rho_i$ (after removing second order terms)

$$
\begin{aligned}
-pd_1[\frac{\pi}{\Theta_0}(\frac{\partial \eta}{\partial Z} + \lambda\Lambda) + \frac{\partial \Phi}{\partial \Theta}\lambda\Lambda - \frac{\partial \eta}{\partial \Theta}\frac{\gamma_0}{L_0}] &= S_{RR} + H(\rho_i) \\
pd_1\frac{\partial \Omega}{\partial \Theta}\lambda\Lambda &= 0 \\
-pd_1(\frac{\partial \Omega}{\partial \Theta}\frac{\gamma_0}{L_0} - \frac{\pi}{\Theta_0}\frac{\partial \Omega}{\partial Z}) &= 0
\end{aligned}
\tag{7.41}
$$

This clearly presents several problems, in particular we would need to make the assumption that the pressure is applied only on the radial normal face. For the Lagrangian multiplier we have that

$$
S_{RR} + H(\rho_i) = -pd_1\frac{\pi}{\Theta_0}\lambda\Lambda \tag{7.42}
$$

hence

$$
K = -D^K_{S_{11}}\frac{C_0}{2}e^{C_1} - pd_1\frac{\pi}{\Theta_0}\lambda\Lambda \tag{7.43}
$$

Additionally there will be an implicit relation where the partial derivative terms from S_{RR} and $\text{proj}_{\hat{R}}\hat{n}_l$ in the first equation of (7.41) must cancel. For $\vec{t} = \frac{N}{A_e^{(1)}}\text{proj}_{\hat{Z}}\hat{n}_l$, where $A_e^{(1)}$ is the cross-sectional surface area of the deformed vessel, we have the following three relations at $Z = 0, L_0$

$$
\begin{aligned}
0 &= S_{ZR} \\
0 &= S_{Z\Theta} \\
\frac{N}{A_e^{(1)}}d_4(\frac{\pi}{\Theta_0} + \frac{\partial \Phi}{\partial \Theta}) &= S_{ZZ} + H(R) + H(\rho_o)
\end{aligned}
\tag{7.44}
$$

The second equation results in an inconsistency, namely $D^K_{S_{23}} = 0$, but this is remedied if we consider torsion to be a traction applied in the circumferential direction to maintain equilibrium. Torsion is typically ignored as a traction for linear elastic materials and the vessel is instead considered to be simply spinning around an axis with rotation defined by a relation for the distribution of the applied moment across the radius. However, energy formulations for torsional work in linear elastic materials would indicate that rotational problems should be able to have a traction formulation for the applied moment. Including this term the total traction vector at the deformed end surfaces will be $\vec{t} = D^K_{S_{32}}\hat{\Theta} + \frac{N}{A_e^{(1)}}\text{proj}_{\hat{Z}}\hat{n_l}$. Note that the additional moment traction drops out upon substitution into the equilibrium equations. Also note the LHS of equation three from (7.44) is equivalent to a negative pressure applied on the Z normal face.

Deformation Theory Normals

We have the following relation at $R = \rho_i$ (after removing high order terms)

$$-pd_3[\frac{\pi}{\Theta_0}(\frac{\partial\eta}{\partial Z} + \lambda\Lambda) + \frac{\partial\Phi}{\partial\Theta}\lambda\Lambda - \frac{\partial\eta}{\partial\Theta}\frac{\gamma_0}{L_0}] = S_{RR} + H(\rho_i) \tag{7.45}$$

Note that this result is the same as using the parametric surface normals if we make the assumption that the pressure is only applied on the radial normal face. In solid mechanics the hydrostatic pressure is generally not assumed to have this type of loading so the result from Nanson's relations is slightly more valid even though the result is the same. Going forward we willl not enforce the condition

$$\frac{\pi}{\Theta_0}\frac{\partial\eta}{\partial Z} + \frac{\partial\Phi}{\partial\Theta}\lambda\Lambda - \frac{\partial\eta}{\partial\Theta}\frac{\gamma_0}{L_0} = 0 \tag{7.46}$$

although it must be true due to the Cauchy boundary conditions. Solving for K from equation (7.45) we obtain

$$K = -D^K_{S_{11}}\frac{C_0}{2}e^{C_1} - pd_3\frac{\pi}{\Theta_0}\lambda\Lambda \tag{7.47}$$

There is a problem that was not addressed in the previous section. If we evaluate $D^K_{S_{11}}\frac{C_0}{2}e^{C_1} + H(\rho_o)$ we obtain a significantly positive pressure at the outside of the vessel, meaning that the pressure is not just slightly distributed through the wall (in fact, it is completely distributed by ρ_0). To rectify this, we must shrink the thickness of the vessel around the median point ρ_0. We shrink the vessel by solving the following equation where $\rho_0 - \epsilon$ is the new interior of the vessel and $\rho_0 + \epsilon$ is the new exterior

$$\ln(\frac{\rho_0 + \epsilon}{\rho_0 - \epsilon})(D^K_{S_{22}} - D^K_{S_{11}})\frac{C_0}{2}e^{C_1} - d_3\frac{\pi}{\Theta_0}\lambda\Lambda p = -\nu p \tag{7.48}$$

Let

$$\ln(x) = \frac{2}{(D^K_{S_{22}} - D^K_{S_{11}})C_0 e^{C_1}} p(d_3\frac{\pi}{\Theta_0}\lambda\Lambda - \nu) \tag{7.49}$$

where $0 \leq \nu \leq 1$ and $(1 - \nu)$ is the fraction of the internal pressure distributed through the wall. Then

$$\epsilon = \rho_0 \frac{x - 1}{x + 1} \tag{7.50}$$

We can then adjust ν as needed. Henceforth we define $\rho_i = \rho_0 - \epsilon$ and $\rho_o = \rho_0 + \epsilon$. This will become important for the energy equation. For $\vec{t} = D^K_{S_{32}}\hat{\Theta} + \frac{N}{A_e^{(1)}}\text{proj}_{\hat{Z}}\hat{n}_l$ we have the following three relations at $Z = 0, L_0$

$$\begin{aligned} 0 &= S_{ZR} \\ D^K_{S_{32}} &= S_{Z\Theta} \\ \frac{N}{A_e^{(1)}} d_4(\frac{\pi}{\Theta_0} + \frac{\partial \Phi}{\partial \Theta}) &= S_{ZZ} + H(R) + H(\rho_o) \end{aligned} \tag{7.51}$$

7.4.3 Cauchy Stress Traction Boundary Conditions

The Cauchy traction-stress equations relate the traction to the true stresses as opposed to the nominal traction-stress equations which relate the traction to the internal forces normalized by the undeformed area. Looking at the Cauchy stress boundary conditions for the internal lateral surface, we have the following (after canceling the normal vectors on both sides and second-order/zero terms)

$$-p = D^K_{F_{11}} D^K_{S_{11}} + H'(\rho_i) \tag{7.52}$$

where $H'(\rho_i)$ is the hydrostatic pressure taken with respect to the deformed area. Solving for $-p$ from both sets of boundary conditions yields that $H'(\rho_i) = D^K_{F_{11}} H(\rho_i)$. Hence $d_3 = \frac{\rho_0}{R_0}$. For the end surfaces we have

$$\begin{aligned} 0 =& \frac{1}{R_0}\frac{\partial \Omega}{\partial \Theta} D^K_{S_{23}} + \frac{\partial \Omega}{\partial Z} D^K_{S_{33}} \\ 0 =& -\frac{\partial \eta}{\partial \Theta}(D^K_{F_{11}} D^K_{S_{22}} + D^K_{F_{23}} D^K_{S_{32}}) + (\frac{\pi}{\Theta_0} + \frac{\partial \Phi}{\partial \Theta})(F_{\Theta\Theta} S_{\Theta Z} + F_{\Theta Z} S_{ZZ}) \\ \frac{N}{A_e^{(1)}}(\frac{\pi}{\Theta_0} + \frac{\partial \Phi}{\partial \Theta}) =& -\frac{\partial \eta}{\partial \Theta}(D^K_{F_{22}} D^K_{S_{23}} + D^K_{F_{23}} D^K_{S_{33}}) + (\frac{\pi}{\Theta_0} + \frac{\partial \Phi}{\partial \Theta})(F_{Z\Theta} S_{\Theta Z} + F_{ZZ} S_{ZZ}) \end{aligned} \tag{7.53}$$

7.4.4 Eigenvalue Compatibility Condition

The first equation of (7.53) gives us a relation not present in the nominal stress boundary conditions, namely that for our particular assumed displacement field we have

$$m = n \frac{L_0}{R_0 \pi} \frac{D^K_{S_{23}}}{D^K_{S_{33}}} \tag{7.54}$$

We can use this condition to exclude buckling modes which are not possible, and it simplifies the last term of the last equation of (7.53) so that $\frac{1}{R_0}\frac{\partial \eta}{\partial \Theta} D^K_{S_{23}} + \frac{\partial \eta}{\partial Z} D^K_{S_{33}}$ from σ_{zz} cancels. Comparing the third equation to the nominal stress counterpart we have

$$H(R) + H(\rho_o) + D^K_{S_{33}} = \frac{\pi}{\Theta_0} d_4 (D^K_{F_{33}} D^K_{S_{33}} + H'(R) + H'(\rho_o)) \tag{7.55}$$

Hence $d_4 = \frac{\Theta_0}{\pi\lambda\Lambda} = D^K_{F_{11}} d_3$ and $H(R) = \frac{1}{\lambda\Lambda} H'(R)$. The second Cauchy relation gives us the same inconsistency that arises from the nominal stress boundary conditions. The Cauchy moment traction M_c will be

$$M_c = \frac{\pi}{\Theta_0} (D^K_{F_{22}} D^K_{S_{23}} + D^K_{F_{23}} D^K_{S_{33}}) \tag{7.56}$$

Equating the forces above with those derived from the nominal stress traction M_n we have

$$\frac{\pi}{\Theta_0} (D^K_{F_{22}} D^K_{S_{23}} + D^K_{F_{23}} D^K_{S_{33}}) A^{(1)}_e = D^K_{S_{32}} A^{(0)}_e \tag{7.57}$$

Hence $M_n = d_4 M_c = \frac{\Theta_0}{\pi\lambda\Lambda} M_c$ and we have a direct correspondence between stress terms that must be met, i.e. $D^K_{F_{22}} D^K_{S_{23}} + D^K_{F_{23}} D^K_{S_{33}} = \lambda\Lambda D^K_{S_{32}}$. This can be confirmed by computation, and note that the moment traction is the applied moment normalized by an unknown volume.

Remark. In a misguided attempt to “cheat death,” so to speak, I attempted to provide a framework with parameters that can be optimized to best fit the Lagrangian multiplier to the data while still maintaining that the derivative is $\frac{1}{R}(D^K_{S_{22}} - D^K_{S_{11}})$ for the original thickness of the vessel. However, of course a function that has the same derivative as the original logarithmic hydrostatic pressure distribution should intersect the same points if a boundary condition is met. The function can only be translated, not dilated since the slope must be maintained, and that is what is happening in the derivations below. I did not see it right away and an equivalent operation would have simply translated the original function. It is reformulating $y = \log(x)$ as $\log(x) = \log(\frac{x}{\prod_{j=1}^N x_j^{\frac{1}{N}}}) + \sum_{j=1}^N y_j$ which is essentially a trivial application of logarithm laws. The issue was solved by shrinking the vessel thickness. Consider

$$H(R) = \frac{1}{N} \sum_{j=1}^{N} H_j \tag{7.58}$$

where

$$H_j = \text{ln}(\frac{R}{\rho_j})(D^K_{S_{22}} - D^K_{S_{11}}) - (D^K_{S_{11}} + \nu p) g_j \tag{7.59}$$

where $\nu = c_0 \frac{\pi}{\Theta_0} \lambda \Lambda$ and $g_j(\rho_j)$ is a discrete function with $0 \leq g_j \leq g_1 = 1$. Hence

$$\sum_{j=1}^{N} H_j = \text{ln}(\frac{R^N}{\prod_{j=1}^{N} \rho_j})(D^K_{S_{22}} - D^K_{S_{11}}) - (D^K_{S_{11}} + \nu p) \sum_{j=1}^{N} g_j \tag{7.60}$$

Then the error at the inner radius is given by

$$E(\rho_i) = \frac{1}{N} \text{ln}(\frac{\rho_i^N}{\prod_{j=1}^{N} \rho_j})(D^K_{S_{22}} - D^K_{S_{11}}) - (D^K_{S_{11}} + \nu p)(\frac{1}{N} \sum_{j=1}^{N} g_j - g_1) \tag{7.61}$$

and in general is given by

$$E(\rho_k) = \frac{1}{N} \text{ln}(\frac{\rho_k^N}{\prod_{j=1}^{N} \rho_j})(D^K_{S_{22}} - D^K_{S_{11}}) - (D^K_{S_{11}} + \nu p)(\frac{1}{N} \sum_{j=1}^{N} g_j - g_k) \tag{7.62}$$

For the total error we have

$$\begin{aligned} \sum_{k=1}^{N} E(\rho_k) &= \sum_{k=1}^{N} \frac{1}{N} \text{ln}(\frac{\rho_k^N}{\prod_{j=1}^{N} \rho_j})(D^K_{S_{22}} - D^K_{S_{11}}) + \sum_{k=1}^{N} g_k (D^K_{S_{11}} + \nu p)(\frac{N}{N} - 1) \\ &= \sum_{k=1}^{N} \frac{1}{N} \text{ln}(\frac{\rho_k^N}{\prod_{j=1}^{N} \rho_j})(D^K_{S_{22}} - D^K_{S_{11}}) \leq N \text{ln}(\frac{\rho_o}{\rho_i})(D^K_{S_{22}} - D^K_{S_{11}}) \end{aligned} \tag{7.63}$$

Let N be odd, then the sum of the error at points $R = \rho_i, \rho_0, \rho_o$ will be

$$E_3 = \ln(\frac{\rho_i \rho_0 \rho_o}{\prod_{j=1}^{N} \rho_j^{\frac{3}{N}}})(D^K_{S_{22}} - D^K_{S_{11}}) - (D^K_{S_{11}} + \nu p)(\frac{3}{N}\sum_{j=1}^{N} g_j - g_1 - g_{\frac{N+1}{2}} - g_N) \tag{7.64}$$

Tending $E_3 \to 0$

$$\frac{3}{N}\sum_{j=1}^{N} g_j = \frac{1}{(D^K_{S_{11}} + \nu p)}\ln(\frac{\rho_i \rho_0 \rho_o}{\prod_{j=1}^{N} \rho_j^{\frac{3}{N}}})(D^K_{S_{22}} - D^K_{S_{11}}) + g_1 + g_{\frac{N+1}{2}} + g_N \tag{7.65}$$

Hence a suitable candidate may be

$$\begin{aligned} H &= \ln(\frac{R}{\prod_{j=1}^{N} \rho_j^{\frac{1}{N}}})(D^K_{S_{22}} - D^K_{S_{11}}) - \frac{1}{3}\ln(\frac{\rho_i \rho_0 \rho_o}{\prod_{j=1}^{N} \rho_j^{\frac{3}{N}}})(D^K_{S_{22}} - D^K_{S_{11}}) - \frac{1}{3}(D^K_{S_{11}} + \nu p)(g_1 + g_{\frac{N+1}{2}} + g_N) \\ &= \frac{1}{3}\ln(\frac{R^3}{\rho_i \rho_0 \rho_o})(D^K_{S_{22}} - D^K_{S_{11}}) - \frac{1}{3}(D^K_{S_{11}} + \nu p)(g_1 + g_{\frac{N+1}{2}} + g_N) \end{aligned} \tag{7.66}$$

and we can simply choose $g_1, g_{\frac{N+1}{2}}, g_N$. Then

$$\frac{\partial H}{\partial R} = \frac{1}{N}\frac{NR^{N-1}}{R^N}(D^K_{S_{22}} - D^K_{S_{11}}) = \frac{1}{R}(D^K_{S_{22}} - D^K_{S_{11}}) \tag{7.67}$$

as desired. Setting $H(\rho_i) := -D^K_{S_{11}} - \nu p$ and solving for $(g_1 + g_{\frac{N+1}{2}} + g_N)$ gives us the same $H(R)$ as using any value of N including $N = 1$.

Deformation Theory Normals

We can now substitute for $S_{ZR}, S_{Z\Theta}, S_{ZZ}$ into the equilibrium equations

$$\begin{aligned} 0 = (D^K_{S_{11}} - D^K_{S_{22}})(C_2\Omega + C_3\Phi^\Theta + C_4\Phi^Z + C_5\eta^\Theta + C_6\eta^Z) + (D^\Omega_{S_{11}} - D^\Omega_{S_{22}})\Omega + D^{\Omega^{22}}_{S^2_{21}}\Omega^{\Theta\Theta} \\ + D^{\Omega^{32}}_{S^2_{21}}\Omega^{\Theta Z} + (D^{\Phi^2}_{S_{11}} - D^{\Phi^2}_{S_{22}})\Phi^\Theta + (D^{\Phi^3}_{S_{11}} - D^{\Phi^3}_{S_{22}})\Phi^Z - D^{\eta^2}_{S_{22}}\eta^\Theta + (D^{\eta^3}_{S_{11}} - D^{\eta^3}_{S_{22}})\eta^Z \end{aligned} \tag{7.68}$$

$$
\begin{aligned}
0 = & \frac{1}{\rho_0}(D^{\Omega^2}_{S^2_{22}} + D^{\Omega^2}_{S_{21}} + D^{e^2}_{S^2_{22}} D^{\Omega^2}_{e^2})\Omega^{\Theta} + \frac{1}{\rho_0}(D^{\Omega^3}_{S_{21}})\Omega^{Z} + \frac{1}{\rho_0}(D^{\Phi^{22}}_{S^2_{22}} + D^{e^2}_{S^2_{22}} D^{\Phi^{22}}_{e^2})\Phi^{\Theta\Theta} \\
& + \frac{1}{\rho_0}(D^{\Phi^{32}}_{S^2_{22}} + + D^{e^2}_{S^2_{22}} D^{\Phi^{32}}_{e^2})\Phi^{\Theta Z} + \frac{1}{\rho_0}(D^{e^2}_{S^2_{22}} D^{\eta^{22}}_{e^2} + D^{\eta^{22}}_{S^2_{22}})\eta^{\Theta\Theta} + \frac{1}{\rho_0}(D^{\eta^{32}}_{S^2_{22}} + D^{e^2}_{S^2_{22}} D^{\eta^{32}}_{e^2})\eta^{\Theta Z} \\
& + \frac{\Theta_0}{\pi}\frac{\partial}{\partial Z}[D^{K}_{S_{32}}]
\end{aligned}
$$

$$
\begin{aligned}
= & \frac{1}{\rho_0}(D^{\Omega^2}_{S^2_{22}} + D^{\Omega^2}_{S_{21}} + D^{e^2}_{S^2_{22}} D^{\Omega^2}_{e^2})\Omega^{\Theta} + \frac{1}{\rho_0}(D^{\Omega^3}_{S_{21}})\Omega^{Z} + \frac{1}{\rho_0}(D^{\Phi^{22}}_{S^2_{22}} + D^{e^2}_{S^2_{22}} D^{\Phi^{22}}_{e^2})\Phi^{\Theta\Theta} \\
& + \frac{1}{\rho_0}(D^{\Phi^{32}}_{S^2_{22}} + + D^{e^2}_{S^2_{22}} D^{\Phi^{32}}_{e^2})\Phi^{\Theta Z} + \frac{1}{\rho_0}(D^{e^2}_{S^2_{22}} D^{\eta^{22}}_{e^2} + D^{\eta^{22}}_{S^2_{22}})\eta^{\Theta\Theta} + \frac{1}{\rho_0}(D^{\eta^{32}}_{S^2_{22}} + D^{e^2}_{S^2_{22}} D^{\eta^{32}}_{e^2})\eta^{\Theta Z}
\end{aligned}
\tag{7.69}
$$

$$
\begin{aligned}
0 = & \frac{1}{\rho_0}[D^{\Omega^2}_{S^2_{23}}\Omega^{\Theta} + D^{\Phi^{22}}_{S^2_{23}}\Phi^{\Theta\Theta} + D^{\Phi^{32}}_{S^2_{23}}\Phi^{Z\Theta} + D^{\eta^{22}}_{S^2_{23}}\eta^{\Theta\Theta} + D^{\eta^{32}}_{S^2_{23}}\eta^{Z\Theta} + D^{e^2}_{S^2_{23}}(D^{\Omega^2}_{e^2}\Omega^{\Theta} + D^{\Phi^{22}}_{e^2}\Phi^{\Theta\Theta} \\
& + D^{\Phi^{32}}_{e^2}\Phi^{Z\Theta} + D^{\eta^{22}}_{e^2}\eta^{\Theta\Theta} + D^{\eta^{32}}_{e^2}\eta^{Z\Theta})] + \frac{\partial}{\partial Z}[\frac{N}{A_e}(\frac{\pi}{\Theta_0} + \frac{\partial\Phi}{\partial\Theta}) - (H(\rho_0) + H(\rho_o))]
\end{aligned}
$$

$$
\begin{aligned}
= & \frac{1}{\rho_0}(D^{\Omega^2}_{S^2_{23}} + D^{e^2}_{S^2_{23}} D^{\Omega^2}_{e^2})\Omega^{\Theta} + \frac{1}{\rho_0}(D^{\Phi^{22}}_{S^2_{23}} + D^{e^2}_{S^2_{23}} D^{\Phi^{22}}_{e^2})\Phi^{\Theta\Theta} + [\frac{N}{A_e} + \frac{1}{\rho_0}D^{\Phi^{32}}_{S^2_{23}} \\
& + \frac{1}{\rho_0}D^{e^2}_{S^2_{23}} D^{\Phi^{32}}_{e^2}]\Phi^{\Theta Z} + \frac{1}{\rho_0}(D^{\eta^{22}}_{S^2_{23}} + D^{e^2}_{S^2_{23}} D^{\eta^{22}}_{e^2})\eta^{\Theta\Theta} + (\frac{1}{\rho_0}D^{\eta^{32}}_{S^2_{23}} + \frac{1}{\rho_0}D^{e^2}_{S^2_{23}} D^{\eta^{32}}_{e^2})\eta^{\Theta Z}
\end{aligned}
\tag{7.70}
$$

Chapter 8: ALTERNATIVE SOLUTION METHODS

In this chapter we outline two alternative solution methods by which we can solve the buckling problem using the first variation approximation. The first method introduces the concept of the modified equilibrium and requires brief mention of the solution method used in our work "Solutions to the First-Order Buckling Equations of a Fung Hyperelastic Cylindrical Shell Subjected to Torsion, Internal Pressure, and Axial Tension." The second utilizes that the static equilibrium eigenvalue problem can be written as a linear transformation $\mathbf{M}^t$ of the buckling displacement amplitudes. Using the kernel of this transformation we can then impose static equilibrium onto $\Delta\Pi$. Note that in both solution methods we can either solve for the critical angle of twist given eigenmodes m and n, as we did in the original problem, or we can solve for the critical loads given an angle of twist and eigenmodes. The former simply requires using the external load-internal stress equations outlined in Chapter 3.

8.1 Modified Equilibrium

We can use the direct evaluation of the energy equation to solve the buckling problem in a second way using the equations of equilibrium combined with the proof of Theorem 5.1.1 that a critical value of the energy potential functional implies the object is in equilibrium. We define the modified equilibrium equation to be the sum of the three coupled partial differential equations of static equilibrium

$$\begin{aligned} 0 = \frac{1}{R}\frac{\partial S_{\Theta R}}{\partial \Theta} + \frac{\partial S_{ZR}}{\partial Z} + \frac{1}{R}(S_{RR} - S_{\Theta\Theta}) + \frac{1}{R}\frac{\partial S_{\Theta\Theta}}{\partial \Theta} + \frac{\partial S_{Z\Theta}}{\partial Z} + \frac{1}{R}(S_{R\Theta} + S_{\Theta R}) \\ + \frac{1}{R}\frac{\partial S_{\Theta Z}}{\partial \Theta} + \frac{\partial S_{ZZ}}{\partial Z} + \frac{1}{R}S_{RZ} \end{aligned} \tag{8.1}$$

Consolidating the equations as above results in a large set of solutions that will not necessarily satisfy the three coupled equations of static equilibrium. Any solutions obtained from the above relation will contain a smaller set of solutions which satisfy the three coupled static equilibrium equations thereby implying that the solutions are weakened. However, it is necessary so that we obtain only two equations over the perturbation amplitudes directly from the equilibrium condition. We rely on the energy equation and its implied static equilibrium equations to hone the eigenvalue solution set when using the determinant of the resulting square matrix as a condition. The additional constraint provided by the static equilibrium determinant should further guarantee the stronger form of the equilibrium is satisfied. In theory we should obtain a reliable solution set if we satisfy the modified equilibrium, the static equilibrium inherent in the energy equation, the static equilibrium determinant constraint, and the boundary conditions. In this case only the set of solutions that satisfy the static equilibrium should remain from the modified equilibrium solution set. Using the eigenvalue compatibility condition outlined in Chapter 7 we can then determine which buckling modes are possible and determine the critical values from those (m, n) pairs.

8.1.1 Static Equilibrium Solution

We will outline the general protocol for solving the static equilibrium eigenvalue problem. In linear elastic theory the equilibrium equations for buckling materials are solved by assuming kinematically admissible deformations for the buckling displacement field and then solving the subsequent system for a relation between eigenvalues and material parameters, perhaps requiring approximations along the way. It has been demonstrated to be an effective method for hyperelastic materials as well.

For linear elastic materials it is a valid assumption that if the vessel is sufficiently long then the ends of the vessel have little impact on the buckling conformation and can therefore be ignored [24]. This greatly simplifies the problem and allows single order trigonometric displacements instead of what would be Fourier series that would necessarily satisfy kinematic boundary conditions. In the case of the above equilibrium equations, a pattern of sine/cosine/cosine or cosine/sine/sine for the radial/circumferential/axial buckling displacements results in three linear equations over the displacement amplitudes. The subsequent solution method hinges on this assumption, and in fact it may be a source of significant instability in the model. Intuitively we expect a hyperelastic material to buckle in a higher-order manner as a superposition of several trigonometric terms of varying degrees, so we may not expect the buckling modes to be accurate even if the quantitative results are. Regardless, the relation between material parameters and eigenmodes should still be meaningful if at least as an approximation or as an average of actual buckling modes.

Since we are taking the entirety of the vessel length and since the vessels are shells with $t/L << 1$, we will assume the following buckling displacement field

$$\begin{aligned} \Omega &= A\sin(\frac{m\pi Z}{L}+n\Theta) \\ \Phi &= B\cos(\frac{m\pi Z}{L}+n\Theta) \\ \eta &= C\cos(\frac{m\pi Z}{L}+n\Theta) \end{aligned} \tag{8.2}$$

Note that the eigenvalue compatibility condition relies heavily on the regressed material parameters. Since an eight parameter regression is subject to wild variations it is possible that the eigenvalue relation implies a negative buckling mode. As such, in our previous work we required the usage of $\pm$ in front of each eigenvalue. For positive m and n we have the following representations

$$\begin{aligned} \Omega^{\Theta} &\sim An & \Phi^{\Theta} &\sim -Bn & \eta^{\Theta} &\sim -Cn \\ \Omega^{Z} &\sim \frac{Am\pi}{L_0} & \Phi^{Z} &\sim \frac{-Bm\pi}{L_0} & \eta^{Z} &\sim \frac{-Cm\pi}{L_0} \\ \Omega^{\Theta\Theta} &\sim -An^2 & \Phi^{\Theta\Theta} &\sim -Bn^2 & \eta^{\Theta\Theta} &\sim -Cn^2 \\ \Omega^{\Theta Z} &\sim -\frac{Amn\pi}{L_0} & \Phi^{\Theta Z} &\sim \frac{-Bmn\pi}{L_0} & \eta^{\Theta Z} &\sim \frac{-Cmn\pi}{L_0} \\ \Omega^{ZZ} &\sim -\frac{Am^2\pi^2}{L_0^2} & \Phi^{ZZ} &\sim \frac{-Bm^2\pi^2}{L_0^2} & \eta^{ZZ} &\sim \frac{-Cm^2\pi^2}{L_0^2} \end{aligned} \tag{8.3}$$

After substituting the displacement field and using the representations above to simplify calculations, the result is a linear system of coupled algebraic equations over the perturbation amplitudes

$$\mathbf{M}\vec{x} = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ \\ M_{21} & M_{22} & M_{23} \\ \\ M_{31} & M_{32} & M_{33} \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{8.4}$$

Since $\vec{x}$ is in the nullspace of **M** we seek a nontrivial kernel as a condition for vessel instability. There is a nontrivial kernel if and only if the determinant is zero, hence a zero determinant guarantees material failure. If the matrix can be row reduced then we can find a more explicit relation between the displacement amplitudes that induce a solution by finding the basis of the kernel. It is possible that the kernel basis spans $\mathbb{R}^2$, implying that a solution is possible for any possible combinations of two of the displacement amplitudes with a restriction only on the remaining amplitude. Since we have a constraint on acceptable eigenvalues there may be constraints on perturbation amplitudes but it is irrelevant for computation and is instead just an interesting question relating to theory.

Static Equilibrium without Boundary Conditions

We will denote the static equilibrium matrix coefficients as M^s_{ij}. Substituting the similarities into the static equilibrium (including hydrostatic pressure) yields

$$
\begin{aligned}
M^s_{11} =& (D^{\Omega}_{S_{11}} - D^{\Omega}_{S_{22}}) + C_2(D^{K}_{S_{11}} - D^{K}_{S_{22}}) - n^2(D^{\Omega^{22}}_{S^2_{21}}) - \frac{mn\pi}{L_0}(D^{\Omega^{32}}_{S^2_{21}} + R_0 D^{\Omega^{23}}_{S^3_{31}}) - \frac{m^2\pi^2}{L_0^2}(R_0 D^{\Omega^{33}}_{S^3_{31}}) \\
M^s_{12} =& -n(D^{\Phi^2}_{S_{11}} - D^{\Phi^2}_{S_{22}} + C_3(D^{K}_{S_{11}} - D^{K}_{S_{22}})) - \frac{m\pi}{L_0}(D^{\Phi^3}_{S_{11}} - D^{\Phi^3}_{S_{22}} + C_4(D^{K}_{S_{11}} - D^{K}_{S_{22}})) \\
M^s_{13} =& -n(-D^{\eta^2}_{S_{22}} + C_5(D^{K}_{S_{11}} - D^{K}_{S_{22}})) - \frac{m\pi}{L_0}(D^{\eta^3}_{S_{11}} - D^{\eta^3}_{S_{22}} + C_6(D^{K}_{S_{11}} - D^{K}_{S_{22}})) \\
M^s_{21} =& \frac{n}{R_0}(D^{\Omega^2}_{S^2_{22}} + D^{\Omega^2}_{S_{21}} + D^{e^2}_{S^2_{22}} D^{\Omega^2}_{e^2}) + \frac{m\pi}{R_0 L_0}(R_0 D^{\Omega^3}_{S^3_{32}} + D^{\Omega^3}_{S_{21}} + R_0 D^{e^3}_{S^3_{32}} D^{\Omega^3}_{e^3}) \\
M^s_{22} =& -\frac{n^2}{R_0}(D^{\Phi^{22}}_{S^2_{22}} + D^{e^2}_{S^2_{22}} D^{\Phi^{22}}_{e^2}) - \frac{mn\pi}{R_0 L_0}(D^{\Phi^{32}}_{S^2_{22}} + R_0 D^{\Phi^{23}}_{S^3_{32}} + D^{e^2}_{S^2_{22}} D^{\Phi^{32}}_{e^2} + R_0 D^{e^3}_{S^3_{32}} D^{\Phi^{23}}_{e^3}) \\
& - \frac{m^2\pi^2}{L_0^2}(D^{\Phi^{33}}_{S^3_{32}} + D^{e^3}_{S^3_{32}} D^{\Phi^{33}}_{e^3}) \\
M^s_{23} =& -\frac{n^2}{R_0}(D^{\eta^{22}}_{S^2_{22}} + D^{e^2}_{S^2_{22}} D^{\eta^{22}}_{e^2}) - \frac{mn\pi}{R_0 L_0}(D^{e^2}_{S^2_{22}} D^{\eta^{32}}_{e^2} + D^{\eta^{32}}_{S^2_{22}} + R_0 D^{\eta^{23}}_{S^3_{32}} + R_0 D^{e^3}_{S^3_{32}} D^{\eta^{32}}_{e^2}) \\
& - \frac{m^2\pi^2}{L_0^2}(D^{\eta^{33}}_{S^3_{32}} + D^{e^3}_{S^3_{32}} D^{\eta^{33}}_{e^3}) \\
M^s_{31} =& \frac{n}{R_0}(D^{\Omega^2}_{S^2_{23}} + D^{e^2}_{S^2_{23}} D^{\Omega^2}_{e^2}) + \frac{m\pi}{R_0 L_0}(R_0 D^{\Omega^3}_{S^3_{33}} + D^{\Omega^3}_{S_{13}} + R_0 D^{e^3}_{S^3_{33}} D^{\Omega^3}_{e^3}) \\
M^s_{32} =& -\frac{n^2}{R_0}(D^{\Phi^{22}}_{S^2_{23}} + D^{e^2}_{S^2_{23}} D^{\Phi^{22}}_{e^2}) - \frac{mn\pi}{R_0 L_0}(D^{\Phi^{32}}_{S^2_{23}} + R_0 D^{\Phi^{23}}_{S^3_{33}} + D^{e^2}_{S^2_{23}} D^{\Phi^{32}}_{e^2} + R_0 D^{e^3}_{S^3_{33}} D^{\Phi^{23}}_{e^3}) \\
& - \frac{m^2\pi^2}{L_0^2}(D^{\Phi^{33}}_{S^3_{33}} + D^{e^3}_{S^3_{33}} D^{\Phi^{33}}_{e^3}) \\
M^s_{33} =& -\frac{n^2}{R_0}(D^{\eta^{22}}_{S^2_{23}} + D^{e^2}_{S^2_{23}} D^{\eta^{22}}_{e^2}) - \frac{mn\pi}{R_0 L_0}(D^{\eta^{32}}_{S^2_{23}} + D^{e^2}_{S^2_{23}} D^{\eta^{32}}_{e^2} + R_0 D^{\eta^{23}}_{S^3_{33}} + R_0 D^{e^3}_{S^3_{33}} D^{\eta^{32}}_{e^3}) \\
& - \frac{m^2\pi^2}{L_0^2}(D^{\eta^{33}}_{S^3_{33}} + D^{e^3}_{S^3_{33}} D^{\eta^{33}}_{e^3})
\end{aligned}
$$

If the determinant equation is nontrivial in the absence of utilizing the traction boundary conditions, it still provides a relation between material parameters, geometric parameters, loading parameters, and the eigenmodes. This is due to the construction of the triaxially loaded thick-wall vessel where stretch ratio and inflated radius are considered known. From these loading parameters we can retroactively compute the corresponding loads assuming the vessel has been fit well to the

Fung model.

If the determinant equation is trivial for a particular set of material parameters then the eigenvalues m, n have no governing law between them. The matrix coefficients themselves are nonzero by inspection and therefore their products are nonzero; numerically it can be shown that the differences of these terms is also nonzero for the experimental specimens used in our previous work. Hence the determinant will be a nontrivial equation and, if the constraint is satisfied, reduces the number of free variables by one in the determinant equation resulting from the traction substitutions. In theory, if an assumption about the traction boundary conditions is invalid then the static equilibrium will still be satisfied assuming the pre-substitution determinant equation is satisfied. However, solving the two coupled pre- and post-substituted determinant equations becomes very computationally intensive and is impractical. We therefore live and die by the assumptions of the traction loading.

If we assume that terms do not cancel out when taking the determinant we can abstract a worst-case picture of the relation m, n must satisfy in terms of functions of γ.

$$0 = M^s_{11}(M^s_{22}M^s_{33} - M^s_{23}M^s_{32}) - M^s_{12}(M^s_{21}M^s_{33} - M^s_{23}M^s_{31}) + M^s_{13}(M^s_{21}M^s_{32} - M^s_{22}M^s_{31}) \quad (8.5)$$

Rewriting each of the three components yields

$$\begin{aligned} M^s_{11}(M^s_{22}M^s_{33} - M^s_{23}M^s_{32}) &= M^s_{11}\sum_{i,j=0}^{i+j=4} q_{ij}(\gamma)m^i n^j = \sum_{i,j=0}^{i+j=6} a_{ij}(\gamma)m^i n^j + \sum_{i,j=0}^{i+j=4} c_{ij}(\gamma)m^i n^j \\ M^s_{12}(M^s_{21}M^s_{33} - M^s_{23}M^s_{31}) &= M^s_{12}\sum_{i,j=0}^{i+j=3} q'_{ij}(\gamma)m^i n^j = -\sum_{i,j=0}^{i+j=4} b_{ij}(\gamma)m^i n^j \\ M^s_{13}(M^s_{21}M^s_{32} - M^s_{22}M^s_{31}) &= M^s_{13}\sum_{i,j=0}^{i+j=3} q''_{ij}(\gamma)m^i n^j = \sum_{i,j=0}^{i+j=3} d_{ij}(\gamma)m^{i+1} n^j \end{aligned} \quad (8.6)$$

Hence we can distill the static equilibrium condition to the following

$$\sum_{i,j=0}^{i+j=6} a_{ij}(\gamma)m^i n^j + \sum_{i,j=0}^{i+j=4} (b_{ij}(\gamma) + c_{ij}(\gamma))m^i n^j + \sum_{i,j=0}^{i+j=3} d_{ij}(\gamma)m^{i+1} n^j = 0 \quad (8.7)$$

Note that this equation is a polynomial with respect to γ and we therefore have to be careful to constrain acceptable roots when solving numerically.

Static Equilibrium with Traction Boundary Conditions

The matrix components M^t_{ij} from the equilibrium equations which include substitutions from the boundary conditions will be the following

$$
\begin{aligned}
M^t_{11} =& (D^{\Omega}_{S_{11}} - D^{\Omega}_{S_{22}}) + C_2(D^K_{S_{11}} - D^K_{S_{22}}) - n^2(D^{\Omega^{22}}_{S^2_{21}}) - \frac{mn\pi}{L_0}[D^{\Omega^{32}}_{S^2_{21}} - \frac{\rho_0\gamma_0}{L_0}(D^K_{S_{11}} + H(\rho_0))] \\
&+ \frac{m^2\pi^2}{L_0^2}[\frac{\rho_0\pi}{R_0\Theta_0}(D^K_{S_{11}} + H(\rho_0))] \\
M^t_{12} =& -n(D^{\Phi^2}_{S_{11}} - D^{\Phi^2}_{S_{22}} + C_3(D^K_{S_{11}} - D^K_{S_{22}})) - \frac{m\pi}{L_0}(D^{\Phi^3}_{S_{11}} - D^{\Phi^3}_{S_{22}} + C_4(D^K_{S_{11}} - D^K_{S_{22}})) \\
M^t_{13} =& -n(-D^{\eta^2}_{S_{22}} + C_5(D^K_{S_{11}} - D^K_{S_{22}})) - \frac{m\pi}{L_0}(D^{\eta^3}_{S_{11}} - D^{\eta^3}_{S_{22}} + C_6(D^K_{S_{11}} - D^K_{S_{22}})) \\
M^t_{21} =& n(D^{\Omega^2}_{S^2_{22}} + D^{\Omega^2}_{S_{21}} + D^{e^2}_{S^2_{22}} D^{\Omega^2}_{e^2}) + \frac{m\pi}{\rho_0 L_0}(D^{\Omega^3}_{S_{21}}) \\
M^t_{22} =& -n^2(D^{\Phi^{22}}_{S^2_{22}} + D^{e^2}_{S^2_{22}} D^{\Phi^{22}}_{e^2}) - \frac{mn\pi}{L_0}(D^{\Phi^{32}}_{S^2_{22}} + D^{e^2}_{S^2_{22}} D^{\Phi^{32}}_{e^2} - \frac{\rho_0\Theta_0}{\pi} D^K_{S_{32}}) \\
M^t_{23} =& -n^2(D^{\eta^{22}}_{S^2_{22}} + D^{e^2}_{S^2_{22}} D^{\eta^{22}}_{e^2}) - \frac{mn\pi}{L_0}[D^{e^2}_{S^2_{22}} D^{\eta^{32}}_{e^2} + D^{\eta^{32}}_{S^2_{22}} + \frac{\rho_0\Theta_0}{\pi}(D^K_{S_{22}} + H(\rho_0)] \\
M^t_{31} =& \frac{n}{\rho_0}(D^{\Omega^2}_{S^2_{23}} + D^{e^2}_{S^2_{23}} D^{\Omega^2}_{e^2}) \\
M^t_{32} =& -\frac{n^2}{\rho_0}(D^{\Phi^{22}}_{S^2_{23}} + D^{e^2}_{S^2_{23}} D^{\Phi^{22}}_{e^2}) - \frac{mn\pi}{L_0}[\frac{1}{\rho_0} D^{\Phi^{32}}_{S^2_{23}} + \frac{1}{\rho_0} D^{e^2}_{S^2_{23}} D^{\Phi^{32}}_{e^2} \\
&+ \frac{\Theta_0}{\pi}(\frac{N}{A_c} - D^K_{S_{33}} - H(\rho_0) - H(\rho_o))] \\
M^t_{33} =& -\frac{n^2}{\rho_0}(D^{\eta^{22}}_{S^2_{23}} + D^{e^2}_{S^2_{23}} D^{\eta^{22}}_{e^2}) - \frac{mn\pi}{\rho_0 L_0}(D^{\eta^{32}}_{S^2_{23}} + D^{e^2}_{S^2_{23}} D^{\eta^{32}}_{e^2} + \frac{\rho_0\Theta_0}{\pi} D^K_{S_{32}})
\end{aligned}
$$

Note that we are using the traction conditions associated with the deformation theory normals. Solving this system yields the critical angle of twist as a function of eigenmodes m and n.

8.1.2 Modified Equilibrium Solution

Property 8.1.1. *If the static equilibrium is satisfied then the modified equilibrium is automatically satisfied.*

Proof. Recall our assumed perturbations in the radial, circumferential, and z-direction respectively

$$\Omega = A\sin(\frac{m\pi Z}{L_0} + n\Theta) \qquad \Phi = B\cos(\frac{m\pi Z}{L_0} + n\Theta) + B \qquad \eta = C\cos(\frac{m\pi Z}{L_0} + n\Theta) \tag{8.8}$$

Recall that the static equilibrium condition $\nabla \cdot S = 0$ results in the following set of three equations in cylindrical coordinates

$$\begin{aligned}
0 &= \frac{\partial S_{RR}}{\partial R} + \frac{1}{R}\frac{\partial S_{\Theta R}}{\partial \Theta} + \frac{\partial S_{ZR}}{\partial Z} + \frac{1}{R}(S_{RR} - S_{\Theta\Theta}) \\
0 &= \frac{\partial S_{R\Theta}}{\partial R} + \frac{1}{R}\frac{\partial S_{\Theta\Theta}}{\partial \Theta} + \frac{\partial S_{Z\Theta}}{\partial Z} + \frac{1}{R}(S_{R\Theta} + S_{\Theta R}) \\
0 &= \frac{\partial S_{RZ}}{\partial R} + \frac{1}{R}\frac{\partial S_{\Theta Z}}{\partial \Theta} + \frac{\partial S_{ZZ}}{\partial Z} + \frac{1}{R}S_{RZ}
\end{aligned} \tag{8.9}$$

Suppose the static equilibrium above is satisfied. If we compute each S_{ij} using (8.8) we can abstract the general form of (8.9) using constants a_i, b_i, and c_i for $i \in \{1, 2, 3, 4, 5, 6\}$

$$\begin{aligned}
(a_1A + b_1B + c_1C)\sin(\frac{m\pi Z}{L_0} + n\Theta) + (a_2A + b_2B + c_2C)\cos(\frac{m\pi Z}{L_0} + n\Theta) = 0 \\
(a_3A + b_3B + c_3C)\sin(\frac{m\pi Z}{L_0} + n\Theta) + (a_4A + b_4B + c_4C)\cos(\frac{m\pi Z}{L_0} + n\Theta) = 0 \\
(a_5A + b_5B + c_5C)\sin(\frac{m\pi Z}{L_0} + n\Theta) + (a_6A + b_6B + c_6C)\cos(\frac{m\pi Z}{L_0} + n\Theta) = 0
\end{aligned} \tag{8.10}$$

where each A, B, C is a perturbation amplitude from (8.8). Each element $\{x_i : x \in \{a, b, c\}, i \in \{1, 2, 3, 5, 6\}\}$ represents a coefficient computed from (8.9) composed of material parameters and is multiplied by its corresponding amplitude for the sake of notation (a_i with A and so on). The result is a set of six equations over A, B, C

$$a_iA + b_iB + c_iC = 0 \tag{8.11}$$

Recall that the modified equilibrium condition in the attempted solution is taken by summing the three equations in (8.9). If we express the modified equilibrium in terms of the same coefficients in (8.10) we have the following two equations

$$(a_1 + a_3 + a_5)A + (b_1 + b_3 + b_5)B + (c_1 + c_3 + c_5)C = K_1$$
$$(a_2 + a_4 + a_6)A + (b_2 + b_4 + b_6)B + (c_2 + c_4 + c_6)C = K_2 \tag{8.12}$$

where K_1 and K_2 are two unknown constants. We use the same coefficients since the modified equilibrium is derived from the static equilibrium and we can therefore predict its general form. We will show that K_1 and K_2 are zero such that the modified equilibrium is satisfied. By assumption the six equations from (8.11) hold. If we rearrange them

$$a_i A = -b_i B - c_i C \tag{8.13}$$

and substituting these into (8.12) we get

$$K_1 = -(b_1 + b_3 + b_5)B - (c_1 + c_3 + c_5)C + (b_1 + b_3 + b_5)B + (c_1 + c_3 + c_5)C = 0$$
$$K_2 = -(b_2 + b_4 + b_6)B - (c_2 + c_4 + c_6)C + (b_2 + b_4 + b_6)B + (c_2 + c_4 + c_6)C = 0 \tag{8.14}$$

Hence the modified equilibrium is satisfied. □

Per above, if the static equilibrium is satisfied we have the following system

$$\begin{bmatrix} M_{11} & M_{12} & M_{13} \\ a_1 + a_3 + a_5 & b_1 + b_3 + b_5 & c_1 + c_3 + c_5 \\ a_2 + a_4 + a_6 & b_2 + b_4 + b_6 & c_2 + c_4 + c_6 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{8.15}$$

But if we use relation (8.13) we obtain

$$M = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ \\ 0 & 0 & 0 \\ \\ 0 & 0 & 0 \end{bmatrix} \tag{8.16}$$

However, this does not imply

$$\begin{aligned} x_1 + x_3 + x_5 &= 0 \\ x_2 + x_4 + x_6 &= 0 \end{aligned} \tag{8.17}$$

for $x \in \{a, b, c\}$.

Proof. We can show this by way of counterexample. Suppose the static equilibrium holds and let $b_i = c_i = i$ and $A = B = C = 1$

$$a_i A = -b_i B - c_i C \implies a_i = -2i \tag{8.18}$$

$$\begin{aligned} a_1 + a_3 + a_5 &= -18 & \qquad a_2 + a_4 + a_6 &= -24 \\ b_1 + b_3 + b_5 &= 9 & \qquad b_2 + b_4 + b_6 &= 12 \\ c_1 + c_3 + c_5 &= 9 & \qquad c_2 + c_4 + c_6 &= 12 \end{aligned} \tag{8.19}$$

Hence the last two rows of M are nonzero. Note that in this example the two rows are linear combinations by construction but it would be easy to find an example without this property. □

A significant question remains. Namely, what if the modified equilibrium is satisfied but the static equilibrium is not? Let us suppose the modified equilibrium is satisfied. Using equation

(8.14) as reference we require

$$(a_1 + a_3 + a_5)A + (b_1 + b_3 + b_5)B + (c_1 + c_3 + c_5)C = 0$$
$$(a_2 + a_4 + a_6)A + (b_2 + b_4 + b_6)B + (c_2 + c_4 + c_6)C = 0 \tag{8.20}$$

By assumption we have the following

$$a_i A + b_i B + c_i C = K_i \tag{8.21}$$

where some K_i can be zero but not all. Since the modified equilibrium is satisfied we know that

$$K_1 + K_3 + K_5 = 0$$
$$K_2 + K_4 + K_6 = 0 \tag{8.22}$$

If we can somehow restrict $K_i \geq 0$ from the equilibrium equations after having satisfed the modified equilibrium then we obtain the static equilibrium automatically. Of course, at that stage we may as well force each K_i to be zero.

Modified Equilibrium with $\Delta\Pi$ Equation

We begin by substituting the displacement representations as above with the added requirement of keeping track of which equations are derived from which trigonometric terms using the following

$$\begin{array}{ccc} \Omega \sim A & \Phi \sim B & \eta \sim C \\ \Omega' \sim \cos & \Phi' \sim \sin & \eta' \sim \sin \\ \Omega'' \sim \sin & \Phi'' \sim \cos & \eta'' \sim \cos \end{array} \tag{8.23}$$

It does not matter which row comes from where as the determinant is zero. If we take the first row from the energy conservation and multiply it by $2mn\pi$ (such that $\det(M') = 2mn\pi\det(M) = 0$)

and if we take the second/third row to be the sine/cosine coefficients respectively from the static equilibrium with applied boundary conditions we obtain the following coefficients after combining terms

$$
\begin{aligned}
M_{11}^m &= -\frac{1}{2}C_2(1-e^{-C_1})C_0e^{C_1} - \frac{2m\pi p\rho_i}{L_0(\rho_o^2-\rho_i^2)}\\
M_{12}^m &= \frac{1}{2}(C_3n + C_4\frac{m\pi}{L_0})C_0e^{C_1} - T\frac{m\pi}{L_0}\\
M_{13}^m &= \frac{1}{2}(C_5n + C_6\frac{m\pi}{L_0})C_0e^{C_1} - N\frac{m\pi}{L_0}\\
M_{21}^m &= M_{11}^t\\
M_{22}^m &= M_{12}^t\\
M_{23}^m &= M_{13}^t\\
M_{31}^m &= M_{21}^t + M_{31}^t\\
M_{32}^m &= M_{22}^t + M_{32}^t\\
M_{33}^m &= M_{23}^t + M_{33}^t
\end{aligned}
\tag{8.24}
$$

If we are supplied γ and N we can solve for the critical torsion

$$
M_{12}^m = \frac{M_{11}^m(M_{22}^mM_{33}^m - M_{23}^mM_{32}^m) + M_{13}^m(M_{21}^mM_{32}^m - M_{22}^mM_{31}^m)}{(M_{21}^mM_{33}^m - M_{23}^mM_{31}^m)}
$$

$$
T_{crit} = \frac{L_0}{2m\pi}[\frac{1}{2}(nC_3 + \frac{m\pi}{L_0}C_4)C_0e^{C_1} - \frac{M_{11}^m(M_{22}^mM_{33}^m - M_{23}^mM_{32}^m) + M_{13}^m(M_{21}^mM_{32}^m - M_{22}^mM_{31}^m)}{(M_{21}^mM_{33}^m - M_{23}^mM_{31})}]
\tag{8.25}
$$

By contrast, in the original eigenvalue problem we aimed to determine the critical angle of twist. In the same vein we can instead substitute the torsion and axial load equations into our system $\mathbf{M}^m$, noting that T and N are tractions

$$
\begin{aligned}
T &= \frac{2}{\rho_o^2 - \rho_i^2} \int_{\rho_i}^{\rho_o} \sigma_{\theta z} \rho^2 \, d\rho \\
N &= \frac{1}{\rho_o^2 - \rho_i^2} \int_{\rho_i}^{\rho_o} 2\sigma_{zz} - \sigma_{rr} - \sigma_{\theta\theta} \, d\rho
\end{aligned} \tag{8.26}
$$

which will give us a system for γ as a function of pressure and the two eigenmodes. Note that since we derived an expression for the moment traction as $D^K_{S_{32}}$ in Chapter 7 we could attempt to substitute this into the energy equation instead of using the equation above. However, the moment traction in the traction boundary conditions is normalized by an unknown volume whereas the traction used in the energy equation is normalized by an area. Therefore we would need to multiply $D^K_{S_{32}}$ by an unknown length, so instead of guessing which length we will leave the expression as it is above.

8.2 Kernel Basis for Static Equilibrium System

In this section we outline a solution method utilizing the basis of the kernel for $\mathbf{M}^t$. Suppose $B = \{\vec{u}\}$ is a one-dimensional basis for the kernel of $\mathbf{M}^t$. Recall our approximation of the first variation of the total potential energy functional (single term displacement assumption)

$$
\begin{aligned}
\Delta\Pi = &\frac{1}{2}[-C_2(1 - e^{-C_1})a_{mn} + (C_3 n + C_4 \frac{m\pi}{L_0})b_{mn} + (C_5 n + C_6 \frac{m\pi}{L_0})c_{mn}]C_0 e^{C_1} \\
&- (2p\frac{\rho_i}{(\rho_o^2 - \rho_i^2)}a_{mn} + T\frac{m\pi}{L_0}b_{mn} + N\frac{m\pi}{L_0}c_{mn})
\end{aligned} \tag{8.27}
$$

Let $\vec{x} = [a_{mn}\ b_{mn}\ c_{mn}]^T$. By definition of the kernel $\mathbf{M}^t\vec{u} = 0$ so that $\vec{x} \in \text{span}\{\vec{u}\}$ when the static equilibrium and boundary conditions are both satisfied. Therefore if $\delta\Pi$ is to satisfy the static equilibrium we require

$$
\begin{aligned}
0 = \frac{1}{2}[-C_2(1-e^{-C_1})u_1 + (C_3 n + C_4\frac{m\pi}{L_0})u_2 + (C_5 n + C_6\frac{m\pi}{L_0})u_3]C_0 e^{C_1} \\
- (2p\frac{\rho_i}{(\rho_o^2-\rho_i^2)}u_1 + T\frac{m\pi}{L_0}u_2 + N\frac{m\pi}{L_0}u_3)
\end{aligned}
\tag{8.28}
$$

Recognizing a relation from the Euler-Lagrange equations we can simplify

$$
[(C_3 n + C_4\frac{m\pi}{L_0})u_2 + (C_5 n + C_6\frac{m\pi}{L_0})u_3]C_0 e^{C_1} = T\frac{m\pi}{L_0}u_2 + N\frac{m\pi}{L_0}u_3 \tag{8.29}
$$

Since we know the material, geometric, and loading parameters we can use the torsion equation and the axial tension equation to substitute for the tractions T and N respectively

$$
\begin{aligned}
T &= \frac{2}{\rho_o^2-\rho_i^2}\int_{\rho_i}^{\rho_o} \sigma_{\theta z}\rho^2\, d\rho \\
N &= \frac{1}{\rho_o^2-\rho_i^2}\int_{\rho_i}^{\rho_o} 2\sigma_{zz} - \sigma_{rr} - \sigma_{\theta\theta}\, d\rho
\end{aligned}
\tag{8.30}
$$

The remaining equation is a function of the critical angle of twist γ and the eigenvalues m, n.

Chapter 9: DISCUSSION

Energy conservation methods are generally useful in mechanical failure theory. Constructing an expression for the conservation of energy in a buckling vessel relating internal and external potential energy has significance in the context of continuum mechanics as well as the calculus of variations. Variational calculus permits some nonlinearity within the strain energy and work terms while still producing coupled linear partial differential equations and is therefore a better approximation to the behavior of a nonlinear system than using elasticity theory to derive linear partial differential equations of equilibrium.

An elliptic equation was derived by constructing the total potential energy functional of the system using a first-order approximation to the Fung strain energy Q and second-order approximation of e^Q. The same equation must hold for both the circumferential and axial buckling displacements, and from these we obtain the radial buckling displacement as a linear combination of the partial derivatives of either of the other two displacement directions. We can therefore obtain analytical solutions for the buckling displacement field which satisfy the Dirichlet boundary conditions. This is in direct contrast to elasticity theory in which the equations are difficult to solve unless a deformation field is prescribed. The second-order approximation of Q yielded similar equations but without explicit coefficients we cannot classify them or use a change of variables effectively without dividing the discriminant into several cases.

As part of our treatment of the potential energy functional of the system we verified that an extremization of the total potential energy functional or a stationary inflection point necessarily induces equilibrium for compressible hyperelastic materials. In minimizing the first variation of the functional, we constructed Π to reference the point at which the loading T and N became constant but did not specify whether it was a neutral or stable equilibrium. We therefore required knowledge of loading parameters such as the angle of twist which were within the pre-buckling range. Because of the particular loading and defined states, any extremizing solutions will necessarily either zero out the contribution of the external forces (to maintain stable or neutral equilibrium), map the vessel from a stable equilibrium to the bifurcation point, or maximize Π to map the vessel into the unstable equilibrium state. We sought functions which would maximize Π so that solutions to the Euler-Lagrange equations would yield buckling configurations. If we were to derive analytical solutions we would need to verify that $\delta^2\Pi < 0$.

Since the variational approach shows promise, future directions should utilize the strain-displacement relations for finite deformations

$$E_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i} + u_{m,i}u_{m,j}) \tag{9.1}$$

We will therefore be able to construct the strain energy density function without any specified loading parameters. A second-order approximation for e^Q will still need to be made, however solutions should yield the buckling conformation of the vessel as only dependent on the Fung material parameters. In this case we would need to verify that the displacement field induced a maximum (unstable equilibrium) by computing the second variation and confirming $\delta^2\Pi < 0$.

Treating the problem without loading parameters as above requires assumptions to be made regarding the loading of torsion and axial tension. Since we are taking the vessel from an undeformed state, we must have an approximate loading curve for both so that we can characterize the external work. Based on experimental data from our previous work the torsional loading was predominantly linear and therefore linear elastic theory can be used to approximate the torsional loading using a linear torsional rigidity. The loading of the axial tension was exponential as a function of the angle of twist, however by definition we are assuming that the system does not extend during regular loading. Hence the contribution of work due to axial tension will be zero during loading other than during the initial stretch. As mentioned earlier, this is an unrealistic assumption since the vessel contracts during twisting. The vessel is held in place during *in vitro* testing, thereby necessitating increasing axial tension and increasing vessel stretch. Hence the axial tension during loading does in fact do work on the vessel.

As an alternative to solutions obtained above we presented two methods which require strict assumptions on the buckling displacement field and the definition of the pre-buckling state. The pre-buckling state using these methods is assumed to be the bifurcation (neutral equilibrium) point. Hence any nonzero perturbations necessarily map the vessel into a buckled state unless the contribution of the deformation to the potential energy functional is zero. Since we seek relations between critical loading parameters and the buckling eigenmodes, we require nontrivial relations between displacements with trivial contributions. By approximating the minimization of the first variation of the total potential energy using a variational Taylor series we were able to derive a relation for the buckling amplitudes and eigenmodes. Using this relation and the equation obtained by summing the equations of the static equilibrium (requiring the use of the same displacement assumption) we were able to construct a linear system solvable using a determinant. In the same vein, if we are able to determine the basis of the kernel of the linear transformation $\mathbf{M}^t$ which characterizes the mapping of the buckling displacement amplitudes during static equilibrium, we can substitute these amplitudes into our approximation so that we obtain a relation between the angle of twist and the buckling eigenmodes.

The instability of the model derived using elasticity theory in our previous work indicates that the single term trigonometric displacement assumption is not valid for nonlinear materials. Additionally, it showed that instituting first-order approximations to all displacements in the strain-energy density function is a poor approximation. Any publications using first-order approximations for nonlinear materials should consider the theory on which nonlinear deformation is based. That is to say that since second-order displacement terms are not negligible for finite deformations, choosing a perturbation that is so small that orders two or greater can be neglected is unrealistic. Future work will therefore primarily focus on variational calculus and the associated Euler-Lagrange equations instead of attempting to improve the solution method used for the equilibrium equations. In addition to allowing degrees of nonlinearity, the Euler-Lagrange equations are a much more elegant formulation of the same problem.

Appendix A: DERIVATIONS

A.1 Pre-buckling Cauchy Stresses

For $\boldsymbol{\sigma} = \mathbf{F}\frac{\partial \Psi}{\partial \mathbf{E}}\mathbf{F}^T$ we have

$$\sigma_{rr} = (\frac{R_0\Theta_0}{\rho_0\pi\lambda\Lambda})^2[\frac{b_1}{2}((\frac{R_0\Theta_0}{\rho_0\pi\lambda\Lambda})^2-1)+\frac{b_6}{2}(\frac{\rho_0^2\gamma^2\Lambda^2}{L^2}+\lambda^2\Lambda^2-1)+\frac{b_4}{2}((\frac{\rho_0\pi}{R_0\Theta_0})^2-1)]C_0e^Q \tag{A.1}$$

$$\begin{aligned}\sigma_{\theta\theta} = &[(\frac{\rho_0\pi}{R_0\Theta_0})^2[\frac{b_2}{2}((\frac{\rho_0\pi}{R_0\Theta_0})^2-1)+\frac{b_5}{2}(\frac{\rho_0^2\gamma^2\Lambda^2}{L^2}+\lambda^2\Lambda^2-1)+\frac{b_4}{2}((\frac{R_0\Theta_0}{\rho_0\pi\lambda\Lambda})^2-1)] \\ &+4b_8(\frac{\rho_0^2\pi\gamma\Lambda^2}{R_0\Theta_0 L})^2]C_0e^Q+\frac{\rho_0^2\gamma^2\Lambda^2}{L^2}(b_2(\frac{\rho_0^2\gamma^2\Lambda^2}{L^2}+\lambda^2\Lambda^2-1)+\frac{b_5}{2}((\frac{\rho_0\pi}{R_0\Theta_0})^2-1) \\ &+\frac{b_6}{2}((\frac{R_0\Theta_0}{\rho_0\pi\lambda\Lambda})^2-1)]C_0e^Q\end{aligned} \tag{A.2}$$

$$\sigma_{zz} = (\lambda\Lambda)^2(\frac{b_2}{2}(\frac{\rho_0^2\gamma^2\Lambda^2}{L^2}+\lambda^2\Lambda^2-1)+\frac{b_5}{2}((\frac{\rho_0\pi}{R_0\Theta_0})^2-1)+\frac{b_6}{2}((\frac{R_0\Theta_0}{\rho_0\pi\lambda\Lambda})^2-1)C_0e^Q \tag{A.3}$$

$$\begin{aligned}\sigma_{\theta z} = &[\frac{\rho_0\pi\lambda\Lambda}{\Theta_0 R_0}b_8(\frac{\rho_0^2\pi\gamma\Lambda^2}{R_0\Theta_0 L})+\frac{\rho_0\gamma\lambda\Lambda^2}{L}(\frac{b_2}{2}(\frac{\rho_0^2\gamma^2\Lambda^2}{L^2}+\lambda^2\Lambda^2-1)+\frac{b_5}{2}((\frac{\rho_0\pi}{R_0\Theta_0})^2-1) \\ &+\frac{b_6}{2}((\frac{R_0\Theta_0}{\rho_0\pi\lambda\Lambda})^2-1)]C_0e^Q\end{aligned} \tag{A.4}$$

A.2 Buckled Strain Energy Density Function

$$\begin{aligned}Q = &b_1E_{RR}^2+b_2E_{\Theta\Theta}^2+b_3E_{ZZ}^2+2b_4E_{RR}E_{\Theta\Theta}+2b_5E_{\Theta\Theta}E_{ZZ}+2b_6E_{ZZ}E_{RR}+b_7(E_{R\Theta}^2+E_{\Theta R}^2) \\ &+b_8(E_{\Theta Z}^2+E_{Z\Theta}^2)+b_9(E_{RZ}^2+E_{ZR}^2)\end{aligned} \tag{A.5}$$

$$
\begin{aligned}
Q_b = {} & \frac{b_1}{4}K_{RR}^2 + \frac{b_2}{4}[\frac{\rho_0^2\pi}{R_0^2\Theta_0}(2\Phi^\Theta + \frac{\pi}{\Theta_0}) + \frac{2\rho_0\pi^2}{R_0^2\Theta_0^2}\Omega - 1]K_{\Theta\Theta} + \frac{b_3}{4}[(\rho_0^2 + 2\Omega\rho_0)\frac{\gamma_0^2}{L_0^2} + 2\frac{\rho_0^2\gamma_0}{L_0}\Phi^Z \\
& + 2\lambda\Lambda\eta^Z + \lambda^2\Lambda^2 - 1]K_{ZZ} + \frac{b_4}{2}K_{RR}^2 + \frac{b_5}{2}K_{\Theta\Theta}[(\rho_0^2 + 2\Omega\rho_0)\frac{\gamma_0^2}{L_0^2} + \\
& + 2\frac{\rho_0^2\gamma_0}{L_0}\Phi^Z + 2\lambda\Lambda\eta^Z + \lambda^2\Lambda^2 - 1] + \frac{b_5}{2}[(2\frac{\rho_0^2\pi}{R_0^2\Theta_0}\Phi^\Theta + 2\frac{\rho_0\pi^2}{R_0^2\Theta_0^2}\Omega]K_{ZZ} \\
& + \frac{b_6}{2}K_{RR}[(\rho_0^2 + 2\Omega\rho_0)\frac{\gamma_0^2}{L_0^2} + 2\frac{\rho_0^2\gamma_0}{L_0}\Phi^Z + 2\lambda\Lambda\eta^Z + \lambda^2\Lambda^2 - 1] + \frac{b_8}{2}[\frac{\rho_0^2}{R_0}(\frac{\gamma_0}{L_0}\Phi^\Theta + \frac{\pi}{\Theta_0}\Phi^Z \\
& + \frac{\pi\gamma_0}{\Theta_0}L_0) + \frac{2\rho_0\pi\gamma_0}{R_0\Theta_0L_0}\Omega + \frac{\lambda\Lambda}{R_0}\eta^\Theta]K_{\Theta Z}
\end{aligned} \tag{A.6}
$$

$$
\begin{aligned}
C_1 &= \frac{b_1}{4}K_{RR}^2 + \frac{b_2}{4}K_{\Theta\Theta}^2 + \frac{b_3}{4}K_{ZZ}^2 + \frac{b_4}{2}K_{RR}K_{\Theta\Theta} + \frac{b_5}{2}K_{\Theta\Theta}K_{ZZ} + \frac{b_6}{2}K_{RR}K_{ZZ} + \frac{b_8}{2}K_{\Theta Z}^2 \\
C_2 &= \frac{b_2}{2}[\frac{2\rho_0\pi^2}{R_0^2\Theta_0^2}]K_{\Theta\Theta} + \frac{b_3}{2}[2\frac{\rho_0\gamma_0^2}{L_0^2}]K_{ZZ} + b_5K_{\Theta\Theta}[2\frac{\rho_0\gamma_0^2}{L_0^2}] + b_5[2\frac{\rho_0\pi^2}{R_0^2\Theta_0^2}]K_{ZZ} \\
&\quad + b_6K_{RR}[2\frac{\rho_0\gamma_0^2}{L_0^2}] + b_8[\frac{2\rho_0\pi\gamma_0}{R_0\Theta_0L_0}]K_{\Theta Z} \\
C_3 &= \frac{b_2}{2}[\frac{2\rho_0^2\pi}{R_0^2\Theta_0}]K_{\Theta\Theta} + b_5[2\frac{\rho_0^2\pi}{R_0^2\Theta_0}]K_{ZZ} + b_8[\frac{\rho_0^2\gamma_0}{R_0L_0}]K_{\Theta Z} \\
C_4 &= \frac{b_3}{2}[2\frac{\rho_0^2\gamma_0}{L_0}]K_{ZZ} + b_5K_{\Theta\Theta}[2\frac{\rho_0^2\gamma_0}{L_0}] + b_6K_{RR}[2\frac{\rho_0^2\gamma_0}{L_0}] + b_8[\frac{\rho_0^2\pi}{R_0\Theta_0}]K_{\Theta Z} \\
C_5 &= b_8\frac{\lambda\Lambda}{R_0}K_{\Theta Z} \\
C_6 &= 2\lambda\Lambda(b_5K_{\Theta\Theta} + \frac{b_3}{2}K_{ZZ} + b_6K_{RR})
\end{aligned} \tag{A.7}
$$

A.3 Strain Energy and External Work

A.3.1 Pre-buckling Strain Energy

$$
\begin{aligned}
2\int_0^{\frac{\pi}{2}}\int_0^{L_0}\int_{\rho_i}^{\rho_o}\frac{1}{2}C_0(e^{C_7}-1)\,R\,dR\,dZ\,d\Theta &= (\rho_o^2-\rho_i^2)\int_0^{\frac{\pi}{2}}\int_0^{L_0}\frac{1}{2}C_0(e^{C_7}-1)\,dZ\,d\Theta \\
&= \frac{1}{2}(\rho_o^2-\rho_i^2)\frac{\pi L_0}{2}C_0(e^{C_7}-1)
\end{aligned} \tag{A.8}
$$

$$2\int_0^{\frac{\pi}{n}}\int_0^{\frac{L_0}{2m}}\int_{\rho_i}^{\rho_o}\frac{1}{2}C_0(e^{C_7}-1)\,R\,dR\,dZ\,d\Theta = (\rho_o^2-\rho_i^2)\int_0^{\frac{\pi}{n}}\int_0^{\frac{L_0}{2m}}\frac{1}{2}C_0(e^{C_7}-1)\,dZ\,d\Theta$$
$$= \frac{1}{2}(\rho_o^2-\rho_i^2)\frac{\pi L_0}{2mn}C_0(e^{C_7}-1) \quad \text{(A.9)}$$

A.3.2 Buckled Strain Energy

$$2\int_0^{\frac{\pi}{2}}\int_0^{L_0}\int_{\rho_i+\Omega}^{\rho_o+\Omega}\frac{1}{2}C_0[e^{C_1}(1+C_2\Omega+C_3\frac{\partial\Phi}{\partial\Theta}+C_4\frac{\partial\Phi}{\partial Z}+C_5\frac{\partial\eta}{\partial\Theta}+C_6\frac{\partial\eta}{\partial Z})-1]\,R\,dR\,dZ\,d\Theta$$

$$= \frac{1}{2}(\rho_o^2-\rho_i^2)C_0e^{C_1}[\frac{\pi L_0}{2}-\frac{\pi L_0}{2}e^{-C_1}-\frac{C_2L_0}{\pi}\sum_{n,m=1}^{\infty}a_{mn}\frac{1}{mn}[\sin(m\pi+\frac{n\pi}{2})-\sin(\frac{n\pi}{2})]$$
$$+\frac{C_3L_0}{\pi}\sum_{n,m=1}^{\infty}b_{mn}\frac{1}{m}[\sin(m\pi+\frac{n\pi}{2})-\sin(\frac{n\pi}{2})]+C_4\sum_{n,m=1}^{\infty}b_{mn}\frac{1}{n}[\sin(m\pi+\frac{n\pi}{2})-\sin(\frac{n\pi}{2})]$$
$$+\frac{C_5L_0}{\pi}\sum_{n,m=1}^{\infty}c_{mn}\frac{1}{m}[\sin(m\pi+\frac{n\pi}{2})-\sin(\frac{n\pi}{2})]+C_6\sum_{n,m=1}^{\infty}c_{mn}\frac{1}{n}[\sin(m\pi+\frac{n\pi}{2})-\sin(\frac{n\pi}{2})]]$$
$$-(\rho_o-\rho_i)\frac{L_0C_0}{\pi}(e^{C_1}-1)\sum_{n,m=1}^{\infty}a_{mn}\frac{1}{mn}[\sin(m\pi+\frac{n\pi}{2})-\sin(\frac{n\pi}{2})]$$

$$= \frac{1}{2}(\rho_o^2-\rho_i^2)C_0e^{C_1}\frac{\pi L_0}{2}(1-e^{-C_1})+\sum_{n,m=1}^{\infty}\frac{1}{2}(\rho_o^2-\rho_i^2)C_0e^{C_1}(-\frac{C_2L_0}{\pi}a_{mn}\frac{1}{mn}+\frac{C_3L_0}{\pi}b_{mn}\frac{1}{m}$$
$$+C_4b_{mn}\frac{1}{n}+\frac{C_5L_0}{\pi}c_{mn}\frac{1}{m}+C_6c_{mn}\frac{1}{n})[\sin(m\pi+\frac{n\pi}{2})-\sin(\frac{n\pi}{2})] \quad \text{(A.10)}$$

$$2\int_0^{\frac{\pi}{n}}\int_0^{\frac{L_0}{2m}}\int_{\rho_i+\Omega}^{\rho_o+\Omega}\frac{1}{2}C_0[e^{C_1}(1+C_2\Omega+C_3\frac{\partial\Phi}{\partial\Theta}+C_4\frac{\partial\Phi}{\partial Z}+C_5\frac{\partial\eta}{\partial\Theta}+C_6\frac{\partial\eta}{\partial Z})-1]\,R\,dR\,dZ\,d\Theta$$

$$\begin{aligned}&=\frac{1}{2}(\rho_o^2-\rho_i^2)C_0e^{C_1}\frac{\pi L_0}{2mn}(1-e^{-C_1})+\frac{1}{2}(\rho_o^2-\rho_i^2)C_0e^{C_1}(-\frac{C_2L_0}{\pi}a_{mn}\frac{1}{mn}+\frac{C_3L_0}{\pi}b_{mn}\frac{1}{m}\\&+C_4b_{mn}\frac{1}{n}+\frac{C_5L_0}{\pi}c_{mn}\frac{1}{m}+C_6c_{mn}\frac{1}{n})[\sin(\frac{\pi}{2}+\pi)-\sin(\frac{\pi}{2})]\end{aligned}\tag{A.11}$$

A.3.3 External Work

Tension Work

$$\begin{aligned}2N\int_0^{\frac{\pi}{2}}\int_{\rho_i}^{\rho_o}\eta\,R\,dR\,d\Theta|_0^{L_0}&=2N\int_0^{\frac{\pi}{2}}\int_{\rho_i}^{\rho_o}\sum_{n,m=0}^{\infty}c_{mn}\cos(\frac{m\pi Z}{L_0}+n\Theta)\,R\,dR\,d\Theta|_0^{L_0}\\&=(\rho_o^2-\rho_i^2)N\sum_{n,m=1}^{\infty}c_{mn}\frac{1}{n}[\sin(m\pi+\frac{n\pi}{2})-\sin(\frac{n\pi}{2})]\end{aligned}\tag{A.12}$$

$$\begin{aligned}2N\int_0^{\frac{\pi}{n}}\int_{\rho_i}^{\rho_o}\eta\,R\,dR\,d\Theta|_0^{\frac{L_0}{2m}}&=2N\int_0^{\frac{\pi}{n}}\int_{\rho_i}^{\rho_o}c_{mn}\cos(\frac{m\pi Z}{L_0}+n\Theta)\,R\,dR\,d\Theta|_0^{\frac{L_0}{2m}}\\&=(\rho_o^2-\rho_i^2)Nc_{mn}\frac{1}{n}[\sin(\frac{\pi}{2}+\pi)-\sin(\frac{\pi}{2})]\end{aligned}\tag{A.13}$$

Torsion Work

$$\begin{aligned}2T\int_0^{\frac{\pi}{2}}\int_{\rho_i}^{\rho_o}\Phi\,R\,dR\,d\Theta|_0^{L_0}&=2T\int_0^{\frac{\pi}{2}}\int_{\rho_i}^{\rho_o}\sum_{n,m=0}^{\infty}b_{mn}\cos(\frac{m\pi Z}{L_0}+n\Theta)\,R\,dR\,d\Theta|_0^{L_0}\\&=(\rho_o^2-\rho_i^2)T\sum_{n,m=1}^{\infty}b_{mn}\frac{1}{n}[\sin(m\pi+\frac{n\pi}{2})-\sin(\frac{n\pi}{2})]\end{aligned}\tag{A.14}$$

$$2T\int_0^{\frac{\pi}{n}}\int_{\rho_i}^{\rho_o}\Phi\,R\,dR\,d\Theta\Big|_0^{\frac{L_0}{2m}} = T\int_0^{\frac{\pi}{n}}\int_{\rho_i}^{\rho_o} b_{mn}\cos(\frac{m\pi Z}{L_0}+n\Theta)\,R\,dR\,d\Theta\Big|_0^{\frac{L_0}{2m}}$$
$$= (\rho_o^2-\rho_i^2)Tb_{mn}\frac{1}{n}[\sin(\frac{\pi}{2}+\pi)-\sin(\frac{\pi}{2})] \quad \text{(A.15)}$$

Pressure Work

$$2p\rho_i\int_0^{\frac{\pi}{2}}\int_0^{L_0}\Omega\,dZ\,d\Theta = 2p\rho_i\int_0^{\frac{\pi}{2}}\int_0^{L_0}\sum_{n,m=1}^{\infty}a_{mn}\sin(\frac{m\pi Z}{L_0}+n\Theta)\,dZ\,d\Theta$$
$$= 2p\rho_i\frac{L_0}{\pi}\sum_{n,m=1}^{\infty}a_{mn}\frac{1}{mn}[\sin(m\pi+\frac{n\pi}{2})-\sin(\frac{n\pi}{2})] \quad \text{(A.16)}$$

$$2p\rho_i\int_0^{\frac{\pi}{n}}\int_0^{\frac{L_0}{2m}}\Omega\,dZ\,d\Theta = 2p\rho_i\int_0^{\frac{\pi}{n}}\int_0^{\frac{L_0}{2m}}a_{mn}\sin(\frac{m\pi Z}{L_0}+n\Theta)\,dZ\,d\Theta$$
$$= 2p\rho_i\frac{L_0}{\pi}a_{mn}\frac{1}{mn}[\sin(\frac{\pi}{2}+\pi)-\sin(\frac{\pi}{2})] \quad \text{(A.17)}$$

A.3.4 Approximation of Extremum

Inserting (6.9, 6.11, 6.13, 6.15, 6.17) into (6.7)

$$
\begin{aligned}
0 = & \frac{1}{2}(\rho_o^2 - \rho_i^2)C_0 e^{C_1}\frac{\pi L_0}{2}(1 - e^{-C_1}) + \sum_{n,m=1}^{\infty}\frac{1}{2}(\rho_o^2 - \rho_i^2)C_0 e^{C_1}(-\frac{C_2 L_0}{\pi}a_{mn}\frac{1}{mn} + \frac{C_3 L_0}{\pi}b_{mn}\frac{1}{m} \\
& + C_4 b_{mn}\frac{1}{n} + \frac{C_5 L_0}{\pi}c_{mn}\frac{1}{m} + C_6 c_{mn}\frac{1}{n})[\sin(m\pi + \frac{n\pi}{2}) - \sin(\frac{n\pi}{2})] \\
& - \frac{1}{2}(\rho_o^2 - \rho_i^2)\frac{\pi L_0}{2}C_0(e^{C_7} - 1) - (\rho_o^2 - \rho_i^2)N\sum_{n,m=1}^{\infty}c_{mn}\frac{1}{n}[\sin(m\pi + \frac{n\pi}{2}) - \sin(\frac{n\pi}{2})] \\
& - (\rho_o^2 - \rho_i^2)T\sum_{n,m=1}^{\infty}b_{mn}\frac{1}{n}[\sin(m\pi + \frac{n\pi}{2}) - \sin(\frac{n\pi}{2})] \\
& - 2p\frac{L_0}{\pi}\rho_i\sum_{n,m=1}^{\infty}a_{mn}\frac{1}{mn}[\sin(m\pi + \frac{n\pi}{2}) - \sin(\frac{n\pi}{2})]
\end{aligned}
$$

$$
\begin{aligned}
= & \frac{1}{2}(\rho_o^2 - \rho_i^2)C_0\frac{\pi L_0}{2}[(e^{C_1} - 1) - (e^{C_7} - 1)] + \sum_{n,m=1}^{\infty}[\frac{1}{2}(\rho_o^2 - \rho_i^2)C_0 e^{C_1}(\frac{C_2 L_0}{\pi}a_{mn}\frac{1}{mn} + \frac{C_3 L_0}{\pi}b_{mn}\frac{1}{m} \\
& + C_4 b_{mn}\frac{1}{n} + \frac{C_5 L_0}{\pi}c_{mn}\frac{1}{m} + C_6 c_{mn}\frac{1}{n}) - (\rho_o^2 - \rho_i^2)N c_{mn}\frac{1}{n} \\
& - (\rho_o^2 - \rho_i^2)T b_{mn}\frac{1}{n} - 2p\rho_i\frac{L_0}{\pi}a_{mn}\frac{1}{mn}][\sin(m\pi + \frac{n\pi}{2}) - \sin(\frac{n\pi}{2})] \qquad \text{(A.18)}
\end{aligned}
$$

Inserting (6.10, 6.12, 6.14, 6.16, 6.18) into (6.7)

$$
\begin{aligned}
0 = &\frac{1}{2}(\rho_o^2-\rho_i^2)\frac{\pi L_0}{2mn}C_0(e^{C_1}-1)+\frac{1}{2}(\rho_o^2-\rho_i^2)C_0e^{C_1}[-\frac{C_2L_0a_{mn}}{mn\pi}[\sin(\frac{\pi}{2}+\pi)-\sin(\frac{\pi}{2})]\\
&+\frac{C_3L_0b_{mn}}{m\pi}[\sin(\frac{\pi}{2}+\pi)-\sin(\frac{\pi}{2})]+\frac{C_4b_{mn}}{n}[\sin(\frac{\pi}{2}+\pi)-\sin(\frac{\pi}{2})]\\
&+\frac{C_5L_0c_{mn}}{m\pi}[\sin(\frac{\pi}{2}+\pi)-\sin(\frac{\pi}{2})]+\frac{C_6c_{mn}}{n}[\sin(\frac{\pi}{2}+\pi)-\sin(\frac{\pi}{2})]]\\
&-\frac{1}{2}(\rho_o^2-\rho_i^2)\frac{\pi L_0}{2mn}C_0(e^{C_7}-1)-(\rho_o^2-\rho_i^2)\frac{Nc_{mn}}{n}[\sin(\frac{\pi}{2}+\pi)-\sin(\frac{\pi}{2})]\\
&-\frac{2p\rho_ia_{mn}}{n}[\sin(\frac{\pi}{2}+\pi)-\sin(\frac{\pi}{2})]-(\rho_o^2-\rho_i^2)\frac{Tb_{mn}}{n}[\sin(\frac{\pi}{2}+\pi)-\sin(\frac{\pi}{2})]
\end{aligned}
$$

$$
\begin{aligned}
= &\frac{1}{2}(\rho_o^2-\rho_i^2)C_0\frac{\pi L_0}{2}[(e^{C_1}-1)-(e^{C_7}-1)]+[\frac{1}{2}(\rho_o^2-\rho_i^2)C_0e^{C_1}(-\frac{C_2L_0}{\pi}a_{mn}\frac{1}{mn}\\
&+\frac{C_3L_0}{\pi}b_{mn}\frac{1}{m}+C_4b_{mn}\frac{1}{n}+\frac{C_5L_0}{\pi}c_{mn}\frac{1}{m}+C_6c_{mn}\frac{1}{n})-(\rho_o^2-\rho_i^2)Nc_{mn}\frac{1}{n}-(\rho_o^2-\rho_i^2)Tb_{mn}\frac{1}{n}\\
&-2p\rho_i\frac{L_0}{\pi}a_{mn}\frac{1}{mn}][\sin(\frac{\pi}{2}+\pi)-\sin(\frac{\pi}{2})] \qquad \text{(A.19)}
\end{aligned}
$$

A.4 Buckled Elasticity Equilibrium Components

In evaluating each term of the hyperelastic equilibrium we have omitted terms from the product rule in the first line of each component that will be zero due to being a product of displacement functions and therefore classified as high order. Note that $S_{R\Theta} = S_{RZ} = 0$ as expected.

$$\begin{aligned}\frac{\partial S_{\Theta R}}{\partial \Theta}\frac{2}{C_0}e^{-Q} &= (2b_2E_{\Theta\Theta} + 2b_4E_{RR} + 2b_5E_{ZZ})\frac{\partial F_{R\Theta}}{\partial \Theta} + 2b_8E_{\Theta Z}\frac{\partial F_{RZ}}{\partial \Theta} \\ &= (b_2K_{\Theta\Theta} + b_4K_{RR} + b_5K_{ZZ})D^{\Omega^2}_{F_{12}}\Omega^{\Theta\Theta} + b_8K_{\Theta Z}D^{\Omega^3}_{F_{13}}\Omega^{Z\Theta} \\ &= D^{\Omega^{22}}_{S^2_{21}}\Omega^{\Theta\Theta} + D^{\Omega^{32}}_{S^2_{21}}\Omega^{Z\Theta} \qquad \text{(A.20)}\end{aligned}$$

$$\begin{aligned}\frac{\partial S_{ZR}}{\partial Z}\frac{2}{C_0}e^{-Q} &= 2b_8E_{\Theta Z}\frac{\partial F_{R\Theta}}{\partial Z} + (2b_3E_{ZZ} + 2b_5E_{\Theta\Theta} + 2b_6E_{RR})\frac{\partial F_{RZ}}{\partial Z} \\ &= b_8K_{\Theta Z}D^{\Omega^2}_{F_{12}}\Omega^{\Theta Z} + (b_3K_{ZZ} + b_5K_{\Theta\Theta} + b_6K_{RR})D^{\Omega^3}_{F_{13}}\Omega^{ZZ} \\ &= D^{\Omega^{23}}_{S^3_{31}}\Omega^{\Theta Z} + D^{\Omega^{33}}_{S^3_{31}}\Omega^{ZZ} \qquad \text{(A.21)}\end{aligned}$$

$$\begin{aligned}S_{RR}\frac{2}{C_0}e^{-Q} &= (2b_1E_{RR} + 2b_4E_{\Theta\Theta} + 2b_6E_{ZZ})F_{RR} \\ &= [b_1K_{RR} + b_4(K_{\Theta\Theta} + f_{\Theta\Theta}) + b_6(K_{ZZ} + f_{ZZ})]\frac{R_0\Theta_0}{\rho_0\pi\lambda\Lambda} \\ &= [b_1K_{RR} + b_4(K_{\Theta\Theta} + D^{\Omega}_{E_{22}}\Omega + D^{\Phi^2}_{E_{22}}\Phi^{\Theta}) + b_6(K_{ZZ} + D^{\Omega}_{E_{33}}\Omega + D^{\Phi^3}_{E_{33}}\Phi^Z + D^{\eta^3}_{E_{33}}\eta^Z)]D^K_{F_{11}} \\ &= D^K_{S_{11}} + D^{\Omega}_{S_{11}}\Omega + D^{\Phi^2}_{S_{11}}\Phi^{\Theta} + D^{\Phi^3}_{S_{11}}\Phi^Z + D^{\eta^3}_{S_{11}}\eta^Z \qquad \text{(A.22)}\end{aligned}$$

$$
\begin{aligned}
S_{\Theta\Theta}\frac{2}{C_0}e^{-Q} &= (2b_2E_{\Theta\Theta}+2b_4E_{RR}+2b_5E_{ZZ})F_{\Theta\Theta}+2b_8E_{\Theta Z}F_{\Theta Z} \\
&= [b_4K_{RR}+b_2(K_{\Theta\Theta}+f_{\Theta\Theta})+b_5(K_{ZZ}+f_{ZZ})]\frac{\rho_0+\Omega}{R_0}(\Phi^{\Theta}+\frac{\pi}{\Theta_0}) \\
&\quad + b_8(K_{\Theta Z}+f_{\Theta Z})(\rho_0+\Omega)(\Phi^{Z}+\frac{\gamma_0}{L_0}) \\
&= (b_4K_{RR}+b_2K_{\Theta\Theta}+b_5K_{ZZ})(D^{K}_{F_{22}}+D^{\Omega}_{F_{22}}\Omega+D^{\Phi^{\Theta}}_{F_{22}}\Phi^{\Theta})+[b_2(D^{\Omega}_{E_{22}}\Omega+D^{\Phi^2}_{E_{22}}\Phi^{\Theta}) \\
&\quad + b_5(D^{\Omega}_{E_{33}}\Omega+D^{\Phi^3}_{E_{33}}\Phi^{Z}+D^{\eta^3}_{E_{33}}\eta^{Z})]D^{K}_{F_{22}}+b_8K_{\Theta Z}(D^{K}_{F_{23}}+D^{\Phi^3}_{F_{23}}\Phi^{Z}+D^{\Omega}_{F_{23}}\Omega) \\
&\quad + b_8(D^{\Omega}_{E_{23}}\Omega+D^{\Phi^2}_{E_{23}}\Phi^{\Theta}+D^{\Phi^3}_{E_{23}}\Phi^{Z}+D^{\eta^2}_{E_{23}}\eta^{\Theta})D^{K}_{F_{23}} \\
&= D^{K}_{S_{22}}+D^{\Omega}_{S_{22}}\Omega+D^{\Phi^2}_{S_{22}}\Phi^{\Theta}+D^{\Phi^3}_{S_{22}}\Phi^{Z}+D^{\eta^2}_{S_{22}}\eta^{\Theta}+D^{\eta^3}_{S_{22}}\eta^{Z} \qquad \text{(A.23)}
\end{aligned}
$$

$$
\frac{\partial S_{\Theta\Theta}}{\partial\Theta}\frac{2}{C_0}e^{-Q} = D^{\Omega^2}_{S^2_{22}}\Omega^{\Theta}+D^{\Phi^{22}}_{S^2_{22}}\Phi^{\Theta\Theta}+D^{\Phi^{32}}_{S^2_{22}}\Phi^{Z\Theta}+D^{\eta^{22}}_{S^2_{22}}\eta^{\Theta\Theta}+D^{\eta^{32}}_{S^2_{22}}\eta^{Z\Theta}+D^{e^2}_{S^2_{22}}\frac{\partial e^{Q}}{\partial\Theta}e^{-Q} \qquad \text{(A.24)}
$$

$$
\begin{aligned}
\frac{\partial S_{Z\Theta}}{\partial Z}\frac{2}{C_0}e^{-Q} &= 2b_8\frac{\partial E_{Z\Theta}}{\partial Z}F_{\Theta\Theta} + 2b_8E_{Z\Theta}\frac{\partial F_{\Theta\Theta}}{\partial Z} + 2b_8E_{Z\Theta}F_{\Theta\Theta}\frac{\partial e^Q}{\partial Z}e^{-Q} \\
&\quad + \frac{\partial}{\partial Z}(2b_3E_{ZZ} + 2b_5E_{\Theta\Theta} + 2b_6E_{RR})F_{\Theta Z} + (2b_3E_{ZZ} + 2b_5E_{\Theta\Theta} + 2b_6E_{RR})\frac{\partial F_{\Theta Z}}{\partial Z} \\
&\quad + (2b_3E_{ZZ} + 2b_5E_{\Theta\Theta} + 2b_6E_{RR})F_{\Theta Z}\frac{\partial e^Q}{\partial Z}e^{-Q} \\
&= b_8\frac{\partial f_{Z\Theta}}{\partial Z}D^K_{F_{22}} + b_8K_{\Theta Z}(D^{\Omega^3}_{F^3_{22}}\Omega^Z + D^{\Phi^{23}}_{F^3_{22}}\Phi^{\Theta Z}) + b_8K_{\Theta Z}D^K_{F_{22}}\frac{\partial e^Q}{\partial Z}e^{-Q} \\
&\quad + (b_3\frac{\partial f_{ZZ}}{\partial Z} + b_5\frac{\partial f_{\Theta\Theta}}{\partial Z})D^K_{F_{23}} + (b_3K_{ZZ} + b_5K_{\Theta\Theta} + b_6K_{RR})(D^{\Omega^3}_{F^3_{23}}\Omega^Z + D^{\Phi^{33}}_{F^3_{23}}\Phi^{ZZ}) \\
&\quad + (b_3K_{ZZ} + b_5K_{\Theta\Theta} + b_6K_{RR})D^K_{F_{23}}\frac{\partial e^Q}{\partial Z}e^{-Q} \\
&= b_8(D^{\Omega^3}_{E^3_{23}}\Omega^Z + D^{\Phi^{33}}_{E^3_{23}}\Phi^{\Theta Z} + D^{\Phi^{33}}_{E^3_{23}}\Phi^{ZZ} + D^{\eta^{23}}_{E^3_{23}}\eta^{\Theta Z})D^K_{F_{22}} + b_8K_{\Theta Z}(D^{\Omega^3}_{F^3_{22}}\Omega^Z + D^{\Phi^{23}}_{F^3_{22}}\Phi^{\Theta Z}) \\
&\quad + b_8K_{\Theta Z}D^K_{F_{22}}\frac{\partial e^Q}{\partial Z}e^{-Q} + b_3(D^{\Omega^3}_{E^3_{33}}\Omega^Z + D^{\Phi^{33}}_{E^3_{33}}\Phi^{ZZ} + D^{\eta^{33}}_{E^3_{33}}\eta^{ZZ})D^K_{F_{23}} \\
&\quad + b_5(D^{\Omega^3}_{E^3_{22}}\Omega^Z + D^{\Phi^{23}}_{E^3_{22}}\Phi^{\Theta Z})D^K_{F_{23}} + (b_3K_{ZZ} + b_5K_{\Theta\Theta} + b_6K_{RR})(D^{\Omega^3}_{F^3_{23}}\Omega^Z + D^{\Phi^{33}}_{F^3_{23}}\Phi^{ZZ}) \\
&\quad + (b_3K_{ZZ} + b_5K_{\Theta\Theta} + b_6K_{RR})D^K_{F_{23}}\frac{\partial e^Q}{\partial Z}e^{-Q} \\
&= D^{\Omega^3}_{S^3_{32}}\Omega^Z + D^{\Phi^{23}}_{S^3_{32}}\Phi^{\Theta Z} + D^{\Phi^{33}}_{S^3_{32}}\Phi^{ZZ} + D^{\eta^{33}}_{S^{23}_{32}}\eta^{\Theta Z} + D^{\eta^{33}}_{S^3_{32}}\eta^{ZZ} + D^{e^3}_{S^3_{32}}\frac{\partial e^Q}{\partial Z}e^{-Q}
\end{aligned}
\tag{A.25}
$$

$$
\begin{aligned}
S_{\Theta R}\frac{2}{C_0}e^{-Q} &= \frac{\partial w}{\partial E_{\Theta R}}F_{RR} + \frac{\partial w}{\partial E_{\Theta\Theta}}F_{R\Theta} + \frac{\partial w}{\partial E_{\Theta Z}}F_{RZ} \\
&= (b_2K_{\Theta\Theta} + b_4K_{RR} + b_5K_{ZZ})D^{\Omega^2}_{F_{12}}\Omega^{\Theta} + b_8K_{\Theta Z}D^{\Omega^3}_{F_{13}}\Omega^Z \\
&= D^{\Omega^2}_{S_{21}}\Omega^{\Theta} + D^{\Omega^3}_{S_{21}}\Omega^Z
\end{aligned}
\tag{A.26}
$$

$$
\begin{aligned}
\frac{\partial S_{\Theta Z}}{\partial \Theta}\frac{2}{C_0}e^{-Q} &= (2b_2E_{\Theta\Theta} + 2b_4E_{RR} + 2b_5E_{ZZ})\frac{\partial F_{Z\Theta}}{\partial \Theta} + 2b_8\frac{\partial E_{\Theta Z}}{\partial \Theta}F_{ZZ} + 2b_8E_{\Theta Z}\frac{\partial F_{ZZ}}{\partial \Theta} \\
&\quad + 2b_8E_{\Theta Z}F_{ZZ}\frac{\partial e^Q}{\partial \Theta}e^{-Q} \\
&= (b_2K_{\Theta\Theta} + b_4K_{RR} + b_5K_{ZZ})D^{\eta^{22}}_{F^2_{32}}\eta^{\Theta\Theta} + b_8\frac{\partial f_{\Theta Z}}{\partial \Theta}D^{K}_{F_{33}} + b_8K_{\Theta Z}D^{\eta^{32}}_{F^2_{33}}\eta^{Z\Theta} \\
&\quad + b_8K_{\Theta Z}D^{K}_{F_{33}}\frac{\partial e^Q}{\partial \Theta}e^{-Q} \\
&= (b_2K_{\Theta\Theta} + b_4K_{RR} + b_5K_{ZZ})D^{\eta^{22}}_{F^2_{32}}\eta^{\Theta\Theta} + b_8K_{\Theta Z}D^{\eta^{32}}_{F^2_{33}}\eta^{Z\Theta} + b_8K_{\Theta Z}D^{K}_{F_{33}}\frac{\partial e^Q}{\partial \Theta}e^{-Q} \\
&\quad + b_8(D^{\Omega^2}_{E^2_{23}}\Omega^{\Theta} + D^{\Phi^{32}}_{E^2_{23}}\Phi^{\Theta\Theta} + D^{\Phi^{32}}_{E^2_{23}}\Phi^{Z\Theta} + D^{\eta^{22}}_{E^2_{23}}\eta^{\Theta\Theta})D^{K}_{F_{33}} \\
&= D^{\Omega^2}_{S^2_{23}}\Omega^{\Theta} + D^{\Phi^{22}}_{S^2_{23}}\Phi^{\Theta\Theta} + D^{\Phi^{32}}_{S^2_{23}}\Phi^{Z\Theta} + D^{\eta^{22}}_{S^2_{23}}\eta^{\Theta\Theta} + D^{\eta^{32}}_{S^2_{23}}\eta^{Z\Theta} + D^{e^2}_{S^2_{23}}\frac{\partial e^Q}{\partial \Theta}e^{-Q}
\end{aligned}
\tag{A.27}
$$

$$
\begin{aligned}
\frac{\partial S_{ZZ}}{\partial Z}\frac{2}{C_0}e^{-Q} &= 2b_8E_{Z\Theta}\frac{\partial F_{Z\Theta}}{\partial Z} + \frac{\partial}{\partial Z}(2b_3E_{ZZ} + 2b_5E_{\Theta\Theta} + 2b_6E_{RR})F_{ZZ} \\
&\quad + (2b_3E_{ZZ} + 2b_5E_{\Theta\Theta} + 2b_6E_{RR})\frac{\partial F_{ZZ}}{\partial Z} \\
&\quad + (2b_3E_{ZZ} + 2b_5E_{\Theta\Theta} + 2b_6E_{RR})F_{ZZ}\frac{\partial e^Q}{\partial Z}e^{-Q} \\
&= b_8K_{\Theta Z}D^{\eta^{23}}_{F^3_{32}}\eta^{\Theta Z} + (b_3\frac{\partial f_{ZZ}}{\partial Z} + b_5\frac{\partial f_{\Theta\Theta}}{\partial Z})D^{K}_{F_{33}} + (b_3K_{ZZ} + b_5K_{\Theta\Theta} + b_6K_{RR})D^{\eta^{33}}_{F^3_{33}}\eta^{ZZ} \\
&\quad + (b_3K_{ZZ} + b_5K_{\Theta\Theta} + b_6K_{RR})D^{K}_{F_{33}}\frac{\partial e^Q}{\partial Z}e^{-Q} \\
&= b_8K_{\Theta Z}D^{\eta^{23}}_{F^3_{32}}\eta^{\Theta Z} + b_3(D^{\Omega^3}_{E^3_{33}}\Omega^{Z} + D^{\Phi^{33}}_{E^3_{33}}\Phi^{ZZ} + D^{\eta^{33}}_{E^3_{33}}\eta^{ZZ})D^{K}_{F_{33}} \\
&\quad + b_5(D^{\Omega^3}_{E^3_{22}}\Omega^{Z} + D^{\Phi^{23}}_{E^3_{22}}\Phi^{\Theta Z})D^{K}_{F_{33}} + (b_3K_{ZZ} + b_5K_{\Theta\Theta} + b_6K_{RR})D^{\eta^{33}}_{F^3_{33}}\eta^{ZZ} \\
&\quad + (b_3K_{ZZ} + b_5K_{\Theta\Theta} + b_6K_{RR})D^{K}_{F_{33}}\frac{\partial e^Q}{\partial Z}e^{-Q} \\
&= D^{\Omega^3}_{S^3_{33}}\Omega^{Z} + D^{\Phi^{23}}_{S^3_{33}}\Phi^{\Theta Z} + D^{\Phi^{33}}_{S^3_{33}}\Phi^{ZZ} + D^{\eta^{23}}_{S^3_{33}}\eta^{\Theta Z} + D^{\eta^{33}}_{S^3_{33}}\eta^{ZZ} + D^{e^3}_{S^2_{33}}\frac{\partial e^Q}{\partial Z}e^{-Q}
\end{aligned}
\tag{A.28}
$$

$$
\begin{aligned}
\frac{\partial e^Q}{\partial \Theta} e^{-Q} &= \frac{\partial}{\partial \Theta}(2b_2 E_{\Theta\Theta} + 2b_3 E_{ZZ} + 2b_4 E_{RR} E_{\Theta\Theta} + 2b_5 E_{\Theta\Theta} E_{ZZ} + 2b_6 E_{ZZ} E_{RR} + 4b_8 E_{Z\Theta}) \\
&= b_2 \frac{\partial f_{\Theta\Theta}}{\partial \Theta} + b_3 \frac{\partial f_{ZZ}}{\partial \Theta} + \frac{b_4}{2} K_{RR} \frac{\partial f_{\Theta\Theta}}{\partial \Theta} + \frac{b_5}{2}(\frac{\partial f_{\Theta\Theta}}{\partial \Theta} K_{ZZ} + K_{\Theta\Theta} \frac{\partial f_{ZZ}}{\partial \Theta}) \\
&\quad + \frac{b_6}{2} \frac{\partial f_{ZZ}}{\partial \Theta} K_{RR} + 2b_8 \frac{\partial f_{Z\Theta}}{\partial \Theta} \\
&= b_2(D^{\Omega^2}_{E^2_{22}} \Omega^\Theta + D^{\Phi^{22}}_{E^2_{22}} \Phi^{\Theta\Theta}) + b_3(D^{\Omega^2}_{E^2_{33}} \Omega^\Theta + D^{\Phi^{32}}_{E^2_{33}} \Phi^{Z\Theta} + D^{\eta^{32}}_{E^2_{33}} \eta^{Z\Theta}) \\
&\quad + \frac{b_4}{2} K_{RR}(D^{\Omega^2}_{E^2_{22}} \Omega^\Theta + D^{\Phi^{22}}_{E^2_{22}} \Phi^{\Theta\Theta}) + \frac{b_5}{2}(D^{\Omega^2}_{E^2_{22}} \Omega^\Theta + D^{\Phi^{22}}_{E^2_{22}} \Phi^{\Theta\Theta}) K_{ZZ} \\
&\quad + \frac{b_5}{2} K_{\Theta\Theta}(D^{\Omega^2}_{E^2_{33}} \Omega^\Theta + D^{\Phi^{32}}_{E^2_{33}} \Phi^{Z\Theta} + D^{\eta^{32}}_{E^2_{33}} \eta^{Z\Theta}) + \frac{b_6}{2}(D^{\Omega^2}_{E^2_{33}} \Omega^\Theta + D^{\Phi^{32}}_{E^2_{33}} \Phi^{Z\Theta} + D^{\eta^{32}}_{E^2_{33}} \eta^{Z\Theta}) K_{RR} \\
&\quad + 2b_8(D^{\Omega^2}_{E^2_{23}} \Omega^\Theta + D^{\Phi^{32}}_{E^2_{23}} \Phi^{\Theta\Theta} + D^{\Phi^{32}}_{E^2_{23}} \Phi^{Z\Theta} + D^{\eta^{22}}_{E^2_{23}} \eta^{\Theta\Theta}) \\
&= D^{\Omega^2}_{e^2} \Omega^\Theta + D^{\Phi^{22}}_{e^2} \Phi^{\Theta\Theta} + D^{\Phi^{32}}_{e^2} \Phi^{Z\Theta} + D^{\eta^{22}}_{e^2} \eta^{\Theta\Theta} + D^{\eta^{32}}_{e^2} \eta^{Z\Theta} \qquad \text{(A.29)}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial e^Q}{\partial Z} e^{-Q} &= \frac{\partial}{\partial Z}(2b_2 E_{\Theta\Theta} + 2b_3 E_{ZZ} + 2b_4 E_{RR} E_{\Theta\Theta} + 2b_5 E_{\Theta\Theta} E_{ZZ} + 2b_6 E_{ZZ} E_{RR} + 4b_8 E_{Z\Theta}) \\
&= b_2 \frac{\partial f_{\Theta\Theta}}{\partial Z} + b_3 \frac{\partial f_{ZZ}}{\partial Z} + \frac{b_4}{2} K_{RR} \frac{\partial f_{\Theta\Theta}}{\partial Z} + \frac{b_5}{2}(\frac{\partial f_{\Theta\Theta}}{\partial Z} K_{ZZ} + K_{\Theta\Theta} \frac{\partial f_{ZZ}}{\partial Z}) \\
&\quad + \frac{b_6}{2} \frac{\partial f_{ZZ}}{\partial Z} K_{RR} + 2b_8 \frac{\partial f_{Z\Theta}}{\partial Z} \\
&= b_2(D^{\Omega^3}_{E^3_{22}} \Omega^Z + D^{\Phi^{23}}_{E^3_{22}} \Phi^{\Theta Z}) + b_3(D^{\Omega^3}_{E^3_{33}} \Omega^Z + D^{\Phi^{33}}_{E^3_{33}} \Phi^{ZZ} + D^{\eta^{33}}_{E^3_{33}} \eta^{ZZ}) \\
&\quad + \frac{b_4}{2} K_{RR}(D^{\Omega^3}_{E^3_{22}} \Omega^Z + D^{\Phi^{23}}_{E^3_{22}} \Phi^{\Theta Z}) + \frac{b_5}{2}(D^{\Omega^3}_{E^3_{22}} \Omega^Z + D^{\Phi^{23}}_{E^3_{22}} \Phi^{\Theta Z}) K_{ZZ} \\
&\quad + \frac{b_5}{2} K_{\Theta\Theta}(D^{\Omega^3}_{E^3_{33}} \Omega^Z + D^{\Phi^{33}}_{E^3_{33}} \Phi^{ZZ} + D^{\eta^{33}}_{E^3_{33}} \eta^{ZZ}) + \frac{b_6}{2}(D^{\Omega^3}_{E^3_{33}} \Omega^Z + D^{\Phi^{33}}_{E^3_{33}} \Phi^{ZZ} + D^{\eta^{33}}_{E^3_{33}} \eta^{ZZ}) K_{RR} \\
&\quad + 2b_8(D^{\Omega^3}_{E^3_{23}} \Omega^Z + D^{\Phi^{33}}_{E^3_{23}} \Phi^{\Theta Z} + D^{\Phi^{33}}_{E^3_{23}} \Phi^{ZZ} + D^{\eta^{23}}_{E^3_{23}} \eta^{\Theta Z}) \\
&= D^{\Omega^3}_{e^3} \Omega^Z + D^{\Phi^{23}}_{e^3} \Phi^{\Theta Z} + D^{\Phi^{33}}_{e^3} \Phi^{ZZ} + D^{\eta^{23}}_{e^3} \eta^{\Theta Z} + D^{\eta^{33}}_{e^3} \eta^{ZZ} \qquad \text{(A.30)}
\end{aligned}
$$

A.5 Coefficients

$$D^{K}_{F_{11}} = \frac{R_0\Theta_0}{\rho_0\pi\lambda\Lambda}$$

$$D^{\Omega^2}_{F_{12}} = \frac{1}{R_0}$$

$$D^{\Omega^3}_{F_{13}} = 1$$

$$D^{K}_{F_{22}} = \frac{\rho_0\pi}{R_0\Theta_0}$$

$$D^{\Omega}_{F_{22}} = \frac{\pi}{R_0\Theta_0}$$

$$D^{\Phi^2}_{F_{22}} = \frac{\rho_0}{R_0}$$

$$D^{K}_{F_{23}} = \rho_0\frac{\gamma_0}{L_0}$$

$$D^{\Omega}_{F_{23}} = \frac{\gamma_0}{L_0}$$

$$D^{\Phi^3}_{F_{23}} = \rho_0$$

$$D^{\eta^2}_{F_{32}} = \frac{1}{R_0}$$

$$D^{K}_{F_{33}} = \lambda\Lambda$$

$$D^{\eta^3}_{F_{33}} = 1$$

$$D^{\Omega}_{E_{22}} = \frac{2\rho_0\pi^2}{R_0^2\Theta_0^2}$$

$$D^{\Phi^2}_{E_{22}} = \frac{2\rho_0^2\pi}{R_0^2\Theta_0}$$

$$D^{\Omega}_{E_{23}} = \frac{2\rho_0\pi\gamma_0}{R_0\Theta_0L_0}$$

$$D^{\Phi^2}_{E_{23}} = \frac{\rho_0^2\gamma_0}{R_0L_0}$$

$$D^{\Phi^3}_{E_{23}} = \frac{\rho_0^2\pi}{R_0\Theta_0}$$

$$D^{\eta^2}_{E_{23}} = \frac{\lambda\Lambda}{R_0}$$

$$D^{\Omega}_{E_{33}} = 2\rho_0\frac{\gamma_0^2}{L_0^2}$$

$$D^{\Phi^3}_{E_{33}} = 2\frac{\rho_0^2\gamma_0}{L_0}$$

$$D^{\eta^3}_{E_{33}} = 2\lambda\Lambda$$

$$\begin{aligned} D^{\Omega^{22}}_{S^2_{21}} &= (b_2K_{\Theta\Theta} + b_4K_{RR} + b_5K_{ZZ})D^{\Omega^2}_{F_{12}} \\ &= \frac{1}{R_0}(b_2K_{\Theta\Theta} + b_4K_{RR} + b_5K_{ZZ}) \end{aligned}$$

$$D^{\Omega^{32}}_{S^2_{21}} = b_8K_{\Theta Z}D^{\Omega^3}_{F_{13}} = b_8K_{\Theta Z}$$

$$D^{\Omega^{23}}_{S^3_{31}} = b_8K_{\Theta Z}D^{\Omega^2}_{F_{12}}$$

$$D^{\Omega^{33}}_{S^3_{31}} = (b_3K_{ZZ} + b_5K_{\Theta\Theta} + b_6K_{RR})D^{\Omega^3}_{F_{13}}$$

$$\begin{aligned} D^{K}_{S_{11}} &= (b_1K_{RR} + b_4K_{\Theta\Theta} + b_6K_{ZZ})D^{K}_{F_{11}} \\ &= (b_1K_{RR} + b_4K_{\Theta\Theta} + b_6K_{ZZ})\frac{R_0\Theta_0}{\rho_0\pi\lambda\Lambda} \end{aligned}$$

$$\begin{aligned} D^{\Omega}_{S_{11}} &= (b_4D^{\Omega}_{E_{22}} + b_6D^{\Omega}_{E_{33}})D^{K}_{F_{11}} \\ &= (2b_4\frac{\rho_0\pi^2}{R_0\Theta_0^2} + 2b_6\frac{\rho_0\gamma_0^2}{L_0^2})\frac{R_0\Theta_0}{\rho_0\pi\lambda\Lambda} \end{aligned}$$

$$D^{\Phi^2}_{S_{11}} = b_4D^{\Phi^2}_{E_{22}}D^{K}_{F_{11}} = 2b_4(\frac{\rho_0^2\pi}{R_0^2\Theta_0})\frac{R_0\Theta_0}{\rho_0\pi\lambda\Lambda}$$

$$D^{\Phi^3}_{S_{11}} = b_6D^{\Phi^3}_{E_{33}}D^{K}_{F_{11}} = 2b_6(\frac{\rho_0^2\gamma_0}{L_0})\frac{R_0\Theta_0}{\rho_0\pi\lambda\Lambda}$$

$$D^{\eta^3}_{S_{11}} = b_6D^{\eta^3}_{E_{33}}D^{K}_{F_{11}} = 2b_6\frac{R_0\Theta_0}{\rho_0\pi}$$

$$\begin{aligned} D^{K}_{S_{22}} &= (b_4K_{RR} + b_2K_{\Theta\Theta} + b_5K_{ZZ})D^{K}_{F_{22}} + b_8K_{\Theta Z}D^{K}_{F_{23}} \\ &= (b_4K_{RR} + b_2K_{\Theta\Theta} + b_5K_{ZZ})\frac{\rho_0\pi}{R_0\Theta_0} + b_8K_{\Theta Z}\frac{\rho_0\gamma_0}{L_0} \end{aligned}$$

$$\begin{aligned} D^{\Omega}_{S_{22}} &= (b_4K_{RR} + b_2K_{\Theta\Theta} + b_5K_{ZZ})D^{\Omega}_{F_{22}} \\ &\quad + (b_2D^{\Omega}_{E_{22}} + b_5D^{\Omega}_{E_{33}})D^{K}_{F_{22}} \\ &\quad + b_8K_{\Theta Z}D^{\Omega}_{F_{23}} + b_8D^{\Omega}_{E_{23}}D^{K}_{F_{23}} \end{aligned}$$

$$D_{S_{22}}^{\Phi^2} = (b_4 K_{RR} + b_2 K_{\Theta\Theta} + b_5 K_{ZZ}) D_{F_{22}}^{\Phi^2} + (b_2 D_{E_{22}}^{\Phi^2}) D_{F_{22}}^{K} + b_8 D_{E_{23}}^{\Phi^2} D_{F_{23}}^{K}$$

$$D_{S_{22}}^{\Phi^3} = b_5 D_{E_{33}}^{\Phi^Z} D_{F_{22}}^{K} + b_8 K_{\Theta Z} D_{F_{23}}^{\Phi^3} + b_8 D_{E_{23}}^{\Phi^3} D_{F_{23}}^{K}$$

$$D_{S_{22}}^{\eta^2} = b_8 D_{E_{23}}^{\eta^2} D_{F_{23}}^{K} = b_8 \frac{\rho_0 \gamma_0 \lambda \Lambda}{R_0 L_0}$$

$$D_{S_{22}}^{\eta^3} = b_5 D_{E_{33}}^{\eta^3} D_{F_{22}}^{K} = 2b_5 \frac{\rho_0 \pi \lambda \Lambda}{R_0 \Theta_0}$$

$$D_{S_{22}^2}^{\Omega^2} = D_{S_{22}}^{\Omega}$$

$$D_{S_{22}^2}^{\Phi^{22}} = D_{S_{22}}^{\Phi^2}$$

$$D_{S_{22}^2}^{\Phi^{32}} = D_{S_{22}}^{\Phi^3}$$

$$D_{S_{22}^2}^{\eta^{22}} = D_{S_{22}}^{\eta^2}$$

$$D_{S_{22}^2}^{\eta^{32}} = D_{S_{22}}^{\eta^3}$$

$$D_{S_{22}^2}^{e^2} = (b_2 K_{\Theta\Theta} + b_4 K_{RR} + b_5 K_{ZZ}) D_{F_{22}}^{K} + b_8 K_{\Theta Z} D_{F_{23}}^{K}$$

$$D_{S_{32}}^{K} = (b_3 K_{ZZ} + b_5 K_{\Theta\Theta} + b_6 K_{RR}) D_{F_{23}}^{K} + b_8 K_{\Theta Z} D_{F_{22}}^{K}$$

$$D_{S_{32}^3}^{\Omega^3} = b_8 D_{E_{23}}^{\Omega} D_{F_{22}}^{K} + b_8 K_{\Theta Z} D_{F_{22}}^{\Omega} + b_3 D_{E_{33}}^{\Omega} D_{F_{23}}^{K} + b_5 D_{E_{22}}^{\Omega} D_{F_{23}}^{K} + (b_3 K_{ZZ} + b_5 K_{\Theta\Theta} + b_6 K_{RR}) D_{F_{23}}^{\Omega}$$

$$D_{S_{32}^3}^{\Phi^{23}} = b_8 D_{E_{23}}^{\Phi^2} D_{F_{22}}^{K} + b_8 K_{\Theta Z} D_{F_{22}}^{\Phi^2} + b_5 D_{E_{22}}^{\Phi^2} D_{F_{23}}^{K}$$

$$D_{S_{32}^3}^{\Phi^{33}} = b_8 D_{E_{23}}^{\Phi^3} D_{F_{22}}^{K} + b_3 D_{E_{33}}^{\Phi^3} D_{F_{23}}^{K} + (b_3 K_{ZZ} + b_5 K_{\Theta\Theta} + b_6 K_{RR}) D_{F_{23}}^{\Phi^3}$$

$$D_{S_{32}^3}^{\eta^{23}} = b_8 D_{E_{23}}^{\eta^2} D_{F_{22}}^{K} = b_8 \frac{\lambda \Lambda}{R_0} \frac{\rho_0 \pi}{R_0 \Theta_0}$$

$$D_{S_{32}^3}^{\eta^{33}} = b_3 D_{E_{33}}^{\eta^3} D_{F_{23}}^{K} = 2b_3 \lambda \Lambda \frac{\rho_0 \pi}{R_0 \Theta_0}$$

$$D_{S_{32}^3}^{e^3} = b_8 K_{\Theta Z} D_{F_{22}}^{K} + (b_3 K_{ZZ} + b_5 K_{\Theta\Theta} + b_6 K_{RR}) D_{F_{23}}^{K} = b_8 \frac{\rho_0^3 \pi^2 \gamma_0}{R_0^2 \Theta_0^2 L_0}$$

$$D_{S_{21}}^{\Omega^2} = (b_2 K_{\Theta\Theta} + b_4 K_{RR} + b_5 K_{ZZ}) D_{F_{12}}^{\Omega}$$

$$D_{S_{21}}^{\Omega^3} = b_8 K_{\Theta Z} D_{F_{13}}^{\Omega}$$

$$D_{S_{23}}^{K} = b_8 K_{\Theta Z} D_{F_{33}}^{K}$$

$$D_{S_{23}^2}^{\Omega^2} = b_8 D_{E_{23}}^{\Omega} D_{F_{33}}^{K} = 2b_8 \frac{\rho_0 \pi \gamma_0 \lambda \Lambda}{R_0 \Theta_0 L_0}$$

$$D_{S_{23}^2}^{\Phi^{22}} = b_8 D_{E_{23}}^{\Phi^2} D_{F_{33}}^{K} = b_8 \frac{\rho_0^2 \gamma_0 \lambda \Lambda}{R_0 L_0}$$

$$D_{S_{23}^2}^{\Phi^{32}} = b_8 D_{E_{23}}^{\Phi^3} D_{F_{33}}^{K} = b_8 \frac{\pi \lambda \Lambda}{\Theta_0}$$

$$D_{S_{23}^2}^{\eta^{22}} = (b_2 K_{\Theta\Theta} + b_4 K_{RR} + b_5 K_{ZZ}) D_{F_{32}}^{\eta^2} + b_8 D_{E_{23}}^{\eta^2} D_{F_{33}}^{K} = (b_2 K_{\Theta\Theta} + b_4 K_{RR} + b_5 K_{ZZ}) \frac{1}{R_0} + b_8 \frac{\lambda^2 \Lambda^2}{R_0}$$

$$D_{S_{23}^2}^{\eta^{32}} = b_8 K_{\Theta Z} D_{F_{33}}^{\eta^3} = b_8 \frac{2\rho_0^2 \gamma_0}{R_0 L_0}$$

$$D_{S_{23}^2}^{e^2} = b_8 K_{\Theta Z} D_{F_{33}}^{K} = b_8 \frac{2\rho_0^2 \gamma_0}{R_0 L_0}$$

$$D_{S_{33}}^{K} = (b_3 K_{ZZ} + b_5 K_{\Theta\Theta} + b_6 K_{RR}) D_{F_{33}}^{K}$$

$$D_{S_{33}^3}^{\Omega^3} = b_3 D_{E_{33}}^{\Omega} D_{F_{33}}^{K} + b_5 D_{E_{22}}^{\Omega} D_{F_{33}}^{K} = 2b_3 \frac{\rho \gamma_0^2 \lambda \Lambda}{L_0^2} + 2b_5 \frac{\rho_0^2 \pi \lambda \Lambda}{R_0^2 \Theta_0}$$

$$D_{S_{33}^3}^{\Phi^{23}} = b_5 D_{E_{22}}^{\Phi^2} D_{F_{33}}^{K} = 2b_5 \frac{\rho_0^2 \pi \lambda \Lambda}{R_0^2 \Theta_0}$$

$$D_{S_{33}^3}^{\Phi^{33}} = b_3 D_{E_{33}}^{\Phi^3} = 2b_3 \frac{\rho_0^2 \gamma_0}{L_0} \lambda \Lambda$$

$$D_{S_{33}^3}^{\eta^{23}} = b_8 K_{\Theta Z} D_{F_{32}}^{\eta^2} = b_8 \frac{\rho_0^2 \pi \gamma_0}{R_0^2 \Theta_0 L_0}$$

$$D_{S_{33}^3}^{\eta^{33}} = b_3 D_{E_{33}}^{\eta^3} D_{F_3 3}^{K} + (b_3 K_{ZZ} + b_5 K_{\Theta\Theta} + b_6 K_{RR}) D_{F_{33}}^{\eta^3} = 2b_3 \lambda^2 \Lambda^2 + (b_3 K_{ZZ} + b_5 K_{\Theta\Theta} + b_6 K_{RR})$$

$$D_{S_{33}^3}^{e^3} = (b_3 K_{ZZ} + b_5 K_{\Theta\Theta} + b_6 K_{RR}) D_{F_{33}}^{K} = (b_3 K_{ZZ} + b_5 K_{\Theta\Theta} + b_6 K_{RR}) \lambda \Lambda$$

$$D_{S_{13}}^{\Omega^3} = 0$$

$$\begin{aligned} D_{e^2}^{\Omega^2} &= b_2 D_{E_{22}}^{\Omega} + b_3 D_{E_{33}}^{\Omega} + \frac{b_4}{2} K_{RR} D_{E_{22}}^{\Omega} + \frac{b_5}{2} D_{E_{22}}^{\Omega} K_{ZZ} + \frac{b_5}{2} K_{\Theta\Theta} D_{E_{33}}^{\Omega} + \frac{b_6}{2} D_{E_{33}}^{\Omega} + 2b_8 D_{E_{23}}^{\Omega} \\ &= 2b_2 \frac{\rho_0^2 \pi}{R_0^2 \Theta_0} + 2b_3 \frac{\rho_0 \gamma_0^2}{L_0^2} + 2b_4 \frac{\Theta_0}{\pi \lambda^2 \Lambda^2} - b_4 \frac{\rho_0^2 \pi}{R_0^2 \Theta_0} + b_5 \frac{\rho_0^3 \pi \gamma_0^2}{R_0^2 \Theta_0 L_0^2} + \frac{b_5}{2} \frac{\pi}{R_0 \Theta_0} K_{ZZ} + 4b_8 \frac{\rho_0 \pi \gamma_0}{R_0 \Theta_0 L_0} \end{aligned}$$

$$\begin{aligned} D_{e^2}^{\Phi^{22}} &= b_2 D_{E_{22}}^{\Phi^2} + \frac{b_4}{2} K_{RR} D_{E_{22}}^{\Phi^2} + \frac{b_5}{2} D_{E_{22}}^{\Phi^2} K_{ZZ} + 2b_8 D_{E_{23}}^{\Phi^2} \\ &= 2b_2 \frac{\rho_0^2 \pi}{R_0^2 \Theta_0} + 2b_4 \frac{\Theta_0}{\pi \lambda^2 \Lambda^2} - b_4 \frac{\rho_0^2 \pi}{R_0^2 \Theta_0} + \frac{b_5}{2} \frac{\rho_0}{R_0} K_{ZZ} + 2b_8 \frac{\rho_0^2 \gamma_0}{R_0 L_0} \end{aligned}$$

$$\begin{aligned} D_{e^2}^{\Phi^{32}} &= b_3 D_{E_{33}}^{\Phi^3} + \frac{b_5}{2} K_{\Theta\Theta} D_{E_{33}}^{\Phi^3} + \frac{b_6}{2} D_{E_{33}}^{\Phi^3} K_{RR} + 2b_8 D_{E_{23}}^{\Phi^3} \\ &= 2b_3 \frac{\rho_0^2 \gamma_0}{L_0} + \frac{b_5}{2} K_{\Theta\Theta} \frac{\rho_0^2 \gamma_0}{L_0} + b_5 \frac{\rho_0^2 \pi}{R_0^2 \Theta_0} \frac{\rho_0^2 \gamma_0}{L_0} + 2b_8 \frac{\rho_0^2 \pi}{R_0 \Theta_0} \end{aligned}$$

$$D_{e^2}^{\eta^{22}} = 2b_8 D_{E_{23}}^{\eta^2}$$

$$\begin{aligned} D_{e^2}^{\eta^{32}} &= b_3 D_{E_{33}}^{\eta^3} + \frac{b_5}{2} K_{\Theta\Theta} D_{E_{33}}^{\eta^3} + \frac{b_6}{2} D_{E_{33}}^{\eta^3} K_{RR} \\ &= 2b_3 \lambda \Lambda + b_5 \lambda \Lambda + b_5 \frac{\rho_0^2 \pi}{R_0^2 \Theta_0} \lambda \Lambda \end{aligned}$$

$$D_{e^3}^{\Omega^3} = D_{e^2}^{\Omega^2}$$

$$D_{e^3}^{\Phi^{23}} = D_{e^2}^{\Phi^{22}}$$

$$D_{e^3}^{\Phi^{33}} = D_{e^2}^{\Phi^{32}}$$

$$D_{e^3}^{\eta^{23}} = D_{e^2}^{\eta^{22}}$$

$$D_{e^3}^{\eta^{33}} = D_{e^2}^{\eta^{32}}$$

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VITA

Ramsey Shadfan was born and raised in San Antonio, Texas and attended Keystone School for elementary, middle, and high school. After obtaining his diploma in 2008 he attended the Fu Foundation School of Engineering and Applied Science at Columbia University. Originally studying chemical engineering, he changed majors to engineering mechanics after being taken with fluid mechanics. He focused his electives on solid and fluid biomechanics and graduated in 2013 with a Bachelor of Science.

After feeling homesick he moved back home and attended a semester of graduate school at the University of Texas at San Antonio (UTSA) in the biomedical engineering program before deciding to take a break from academics. He worked for his parents at SA Scientific as a mechanical engineer tasked with the development and design of electronic readers for medical diagnostic lateral flow assays. In 2015 he was eager to return to his graduate studies at UTSA and elected to take higher level mathematics classes on the side while continuing his Master's degree in biomedical engineering with a focus on biomechanics.

Having been quite taken with the mathematics courses and professors at UTSA he enrolled in the Master's program for mathematics while working on his research for the biomedical engineering degree. He chose to work on the theoretical modeling of the torsional buckling of a nonlinear cylinder as it had significant ties to higher level mathematics, serving to bridge his academic interests. After reaching out to Dr. Changfeng Gui from the mathematics department for assistance during his research he decided to stay at UTSA for his Ph.D. in the electrical engineering program with Dr. Gui as his advisor. He eagerly looks forward to continuing his graduate studies at UTSA and delving into more sophisticated partial differential equation theory as pertaining to mechanical failure and image processing.