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Impulsive Fractional Cohen-Grossberg Neural Networks: Almost Periodicity Analysis

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Abstract: In this paper, a fractional-order Cohen–Grossberg-type neural network with Caputo fractional derivatives is investigated. The notion of almost periodicity is adapted to the impulsive generalization of the model. General types of impulsive perturbations not necessarily at fixed moments are considered. Criteria for the existence and uniqueness of almost periodic waves are proposed. Furthermore, the global perfect Mittag–Leffler stability notion for the almost periodic solution is defined and studied. In addition, a robust global perfect Mittag–Leffler stability analysis is proposed. Lyapunov-type functions and fractional inequalities are applied in the proof. Since the type of Cohen–Grossberg neural networks generalizes several basic neural network models, this research contributes to the development of the investigations on numerous fractional neural network models.

Keywords: fractional-order derivatives; impulses; Cohen–Grossberg neural networks; almost periodicity; perfect Mittag–Leffler stability; robustness

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1. Introduction

Recently, fractional-order differential systems have attracted a lot of attention in research since fractional-order derivatives are distinguished by its substantial degree of reliability and accuracy. In fact, the mathematical models with fractional-order derivatives are mostly applied in the description of universal laws. This is due to the fact that compared with the classical integer-order derivatives most fractional-order derivatives are non-local and possess memory effects and hereditary properties [1–3].

Among the numerous proposed fractional operators [4,5], one of the most commonly used fractional-order derivative is the Caputo-type derivative. The main reason of such intensive implementation in mathematical modeling is that it has all the advantages of fractional-order derivatives, and in addition, the initial conditions of fractional-order models with Caputo fractional operators can be physically interpreted as in the integer-order models. Hence, there are also advantages related to the geometric interpretations [6–8].

Since more and more experimental results show that real-world models follow fractional calculus dynamics, very recently fractional-order differential systems are successfully applied in various fields of science and engineering [9,10], including COVID-19 models [11].

One of the implications of fractional-order systems is in the neural network modeling. Due to the fact that fractional models are more effective than integer-order models in numerous applications, the existing theory of integer-order neural networks and related models have been extended and improved to the fractional-order case. See, for example, some very resent results in [12–15].

However, the research on the theory and applications of the Cohen–Grossberg-type neural networks with fractional-order derivatives still needs more development. Recently, there are very few results on the existence and stability of their equilibrium states [16–19]. The study of more qualitative properties of such fractional neural network models attracts the interest of the researchers. In this research, we plan to contribute to this idea. In fact, the integer-order Cohen–Grossberg neural networks introduced in [20] have impressive applications in global pattern formation, signal transmission, partial memory storage, pattern recognition, and optimization [21–24]. Fractional-order generalizations of such models that have recently gained great scientific interest have all mentioned the above applications and, at the same time, have the advantages of using fractional-order derivatives. In addition, they generalize a number of fractional cellular neural network models, including fractional Hopfield-type neural network models.

Many researchers also investigated the impulsive effects on the fundamental and qualitative properties of fractional-order models [25–27], including numerous fractional neural network models [28–31]. The huge number of results on impulsive fractional neural networks is further evidence of their remarkable importance for theories and applications. It is well known that impulsive systems and impulsive control strategies have many advantages in the modeling of real-world phenomena that are subject to or can be controlled by short-term perturbations at some instants of time [32–37]. Impulsive generalizations of integer-order Cohen–Grossberg neural networks are also very well studied [38–46].

However, to the best of the authors' knowledge, there are only a few corresponding results on the impulsive generalizations of fractional Cohen–Grossberg neural networks reported in the existing literature. Very recently, Yang, S. et al. [27] presents an example of an impulsive control for the exponential synchronization of fractional Cohen–Grossberg neural networks. In the paper [47], a class of impulsive control fractional order memristive Cohen–Grossberg neural networks with state feedback is introduced, and a synchronization analysis is conducted. The main purpose of this study is to contribute to the development of the theory of almost periodicity for such classes of impulsive fractional neural network models.

The reason for the investigation of almost periodic properties of systems lies in their significance in applied problems where pure periodicity is not applicable. For example, when working with parameters in an almost periodic environment, with repeating solutions with different periods, with seasonal effects with almost periodic behavior, etc. The almost periodicity is a generalization of the periodicity notion and is considered one of the most important qualitative properties of a system [48–50]. The almost periodic behavior of states has been relatively well studied for integer-order systems under impulsive perturbations [51–53]. There are also results on almost periodic properties for some fractional-order systems without impulses [54–56], as well as for impulsive fractional-order systems [57–61]. It should also be noted that the authors in [56] proved that pure periodic solutions do not exist for fractional-order systems.

For integer-order Cohen–Grossberg neural networks under impulsive perturbations, the almost periodicity is studied in very few papers [38,40,45,46]. Our idea is to extend these results to the fractional-order case.

Motivated by the above considerations, the rest of the manuscript is organized according to the following plan. Section 2 is a preliminary section, where the class of impulsive Cohen–Grossberg fractional neural networks is introduced. The considered impulsive perturbations are at variable times since this is the most general case that includes the case of fixed impulsive perturbations and is more relevant to reality [62–64]. The notion of almost periodic solutions is adapted for this class of fractional-order neural networks. Fundamental definitions and lemmas are also presented. In Section 3, the main existence and uniqueness results about almost periodic solutions of the considered impulsive fractional Cohen–Grossberg neural network model are established. Furthermore, the concept of global perfect Mittag–Leffler stability is introduced and considered. For completeness, a model with uncertain parameters is considered. In fact, uncertainties can always affect the qualitative behavior of the solutions [65–67]. Hence, their effects are worth studying. In addition, criteria for the global robust perfect Mittag–Leffler stability of the almost periodic states are proposed. Section 4 is devoted to examples through which we demonstrate the obtained results. The conclusion remarks are stated in Section 5.

Notations. In this paper, $\mathbb{Z}$ stands for the set of all integers; $\mathbb{R}^n$ denotes the n-dimensional Euclidean space; $\mathbb{R}_+ = [0, \infty)$; for a $q \in \mathbb{R}^n$, $q = (q_1, q_2, \ldots, q_n)^T$, we will consider the following norm $||q|| = \sqrt{\sum_{i=1}^n q_i^2}$.

2. Preliminaries

In this section, we will give some basic definitions and lemmas. The fractional-order neural network model under consideration will be also formulated.

2.1. Fractional Calculus Notes

First, we will recall nomenclatures related to fractional calculus.

Definition 1 ([2,3])**.** *Let $t_0 \in \mathbb{R}$. The Caputo fractional derivative of order ν, $0 < \nu < 1$ with a lower limit t_0 for a continuously differentiable in $\mathbb{R}$ real function $\lambda(t)$ is defined by*

$$ {}_{t_0}^{C}D_t^{\nu}\lambda(t) = \frac{1}{\Gamma(1-\nu)} \int_{t_0}^{t} \frac{\lambda'(\sigma)}{(t-\sigma)^{\nu}} d\sigma, $$

where Γ denotes the Gamma function defined as $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$. In particular, if $t_0 = 0$,

$$ {}^{C}D_t^{\nu}\lambda(t) = {}_{0}^{C}D_t^{\nu}\lambda(t) = \frac{1}{\Gamma(1-\nu)} \int_{0}^{t} \frac{\lambda'(\sigma)}{(t-\sigma)^{\nu}} d\sigma. $$

According to [3], the following properties for the Caputo fractional derivatives hold for any $t > t_0$, $t_0 \in \mathbb{R}$:

P1. $I_{t_0}^{\nu}\left({}_{t_0}^{C}D_t^{\nu}\lambda(t)\right) = \lambda(t) - \lambda(t_0)$, where

$$ I_{t_0}^{\nu} = \frac{1}{\Gamma(\nu)} \int_{t_0}^{t} \frac{\lambda(\sigma)}{(t-\sigma)^{1-\nu}} d\sigma $$

is the Riemann–Liouville fractional integral of order $0 < \nu < 1$ with the lower limit t_0;

P2. ${}_{t_0}^{C}D_t^{\nu}(c_1\lambda_1(t) + c_2\lambda_2(t)) = c_1 {}_{t_0}^{C}D_t^{\nu}\lambda_1(t) + c_2 {}_{t_0}^{C}D_t^{\nu}\lambda_2(t)$, $0 < \nu < 1$.

The next property is given in [68].

P3. $\frac{1}{2} {}_{t_0}^{C}D_t^{\nu}(\lambda^2(t)) \leq \lambda(t) {}_{t_0}^{C}D_t^{\nu}\lambda(t)$, $t \geq t_0$, $0 < \nu < 1$.

In the next section, we will also use the class of Mittag–Leffler functions defined as follows [3].

Definition 2. *The standard Mittag–Leffler function is given as*

$$ E_{\nu}(z) = \sum_{\kappa=0}^{\infty} \frac{z^{\kappa}}{\Gamma(\nu\kappa + 1)}, $$

where $\nu > 0$.

Definition 3. *The Mittag–Leffler function with two parameters is defined as*

$$ E_{\nu,\tilde{\nu}}(z) = \sum_{\kappa=0}^{\infty} \frac{z^{\kappa}}{\Gamma(\nu\kappa + \tilde{\nu})}, $$

where $\nu > 0$, $\tilde{\nu} > 0$.

2.2. Model Formulation

Now, we will introduce the impulsive fractional-order Cohen–Grossberg neural network model under consideration.

In [27], the authors considered an impulsive control strategy with impulsive perturbations at fixed instants, applied to the following fractional-order Cohen–Grossberg neural network model

$$ {}_{t_0}^{C}D_t^{\nu} q_i(t) = -a_i(q_i(t)) \left[b_i(q_i(t)) - \sum_{j=1}^{n} c_{ij} f_j(q_j(t)) - I_i(t) \right], \tag{1} $$

where $0 < \nu < 1$, $i = 1, 2, \ldots, n$, n denotes the number of units in the neural network, $q_i(t)$ is the state of the ith unit at time t, $a_i(q_i(t))$ denotes a standard amplification function, $b_i(q_i(t))$ stands for a well-behaved function, $f_j(.)$ stands for the activation function, c_{ij} is the connection weight between the jth neuron and ith neuron, and I_i is the input from outside of the network.

In this paper, we will extend the model proposed in [27], considering variable impulsive perturbations. Some of the parameters will also be generalized. Let $\theta_0(q) = t_0$ for $q \in \mathbb{R}^n$, and the continuous functions $\theta_k : \mathbb{R}^n \to \mathbb{R}$, $k = \pm 1, \pm 2, \ldots$ are such that

$$ \theta_{k-1}(q) < \theta_k(q),\ k \in \mathbb{Z}, \qquad \theta_k(q) \to \pm\infty \text{ as } k \to \pm\infty $$

uniformly on $q \in \mathbb{R}^n$.

In this manuscript, we consider the following fractional-order Cohen–Grossberg neural network model with variable impulsive perturbations:

$$ \begin{cases} {}_{t_0}^{C}D_t^{\nu} q_i(t) = -a_i(q_i(t)) \left[b_i(t, q_i(t)) - \sum_{j=1}^{n} c_{ij}(t) f_j(q_j(t)) - I_i(t) \right], & t \neq \theta_k(q), \\ q_i(t^+) = q_i(t) + p_{ik}(q_i(t)),\ t = \theta_k(q),\ k = \pm 1, \pm 2, \ldots, \end{cases} \tag{2} $$

where the model functions $a_i \in C[\mathbb{R}, \mathbb{R}_+]$, c_{ij}, f_j, I_i, $p_{ik} \in C[\mathbb{R}, \mathbb{R}]$ and $b_i \in C[\mathbb{R}^2, \mathbb{R}_+]$, $i, j = 1, \ldots, n$.

The second condition in Equation (2) is the impulsive condition. The impulsive functions p_{ik} can be used to control the qualitative behavior of the model. Their choice determines the controlled outputs $q_i(t^+)$, $i = 1, 2, \ldots, n$.

The initial condition for Equation (2) is in the form

$$ q(t_0) = q_0, \tag{3} $$

where $q_0 = (q_{10}, q_{20}, \ldots, q_{n0}) \in \mathbb{R}^n$.

We will denote by $q(t) = q(t; t_0, q_0) = (q_1(t; t_0, q_0), q_2(t; t_0, q_0), \ldots, q_n(t; t_0, q_0))^T$ the solution of Equation (2) that satisfies the initial condition from Equation (3).

We denote by

$$ \tau_k :\ t = \theta_k(q),\ q \in \mathbb{R}^n,\ k = \pm 1, \pm 2, \ldots $$

and by t_{l_k}, the moment when the integral curve $(t, q(t))$ of the solution $q(t)$ of Equation (2), Equation (3) meets the hypersurfaces τ_k, i.e., each of the points t_{l_k} is a solution of one of the equations $t = \theta_k(q(t))$. The impulsive points t_{l_k} are the points of discontinuity of the solution $q(t)$ at which

$$ q(t^-) = q(t),\ q(t^+) = q(t^-) + p_k(q(t^-)), $$

where the matrices $p_k = diag(p_{1k}, p_{2k}, \ldots, p_{nk})$, $k = \pm 1, \pm 2, \ldots$. It is also known [25,38,39,42,62,64] that, in general, $k \neq l_k$, $k, l_k = \pm 1, \pm 2, \ldots$ or it is possible for the integral curve $(t, q(t))$ of Equation (2) to not meet the hypersurface τ_k at the moment t_k.

Remark 1. *Equation (2) extends the existing impulsive fractional-order Cohen–Grossberg neural network models [27,47] introducing variable impulsive perturbations. In addition, the neural network model in Equation (2) generalizes numerous impulsive integer-order Cohen–Grossberg* *neural network models investigated in [38–46] to the fractional-order case. Hence, the proposed model (Equation (2)) is not studied before in the existing literature.*

Denote $q(t) = (q_1(t), q_2(t), \dots, q_n(t))^T$, $F(t,q) = (F_1(t,q), F_2(t,q), \dots, F_n(t,q))$,

$$F_i(t,q) = -a_i(q_i(t))\left[b_i(t,q_i(t)) - \sum_{j=1}^{n} c_{ij}(t) f_j(q_j(t)) - I_i(t)\right],$$

where $i = 1, 2, \dots, n$, and Equation (2) can be represented as

$$\begin{cases} {}_{t_0}^{C}D_t^{\nu} q(t) = F(t, q(t)), \quad t \neq \theta_k(q), \\ q(t^+) = q(t) + p_k(q(t)),\ t = \theta_k(q),\ k = \pm 1, \pm 2, \dots. \end{cases} \tag{4}$$

It is well known that considering variable impulsive perturbations is more general and leads to numerous difficulties in existence, uniqueness, and continuability of solutions, such as phenomenon "beating" of solutions, merging solutions after an impulsive perturbations, bifurcation, etc. [25,38,39,42,62]. In order to avoid such complications and, also, to establish our main results on the almost periodicity of solutions, we will assume that $t_{l_k} < t_{l_{k+1}} < \dots$ for $l_k \in \mathbb{Z}$. In addition, we assume that the integral curves of Equation (2) meet each hypersurface τ_k at most once.

2.3. Almost Periodicity Definitions

In this subsection, we will adopt the almost periodicity definitions from [52,53] to the impulsive fractional neural network model (Equation (2)).

First, we will state the following classical definition [49,53].

Definition 4. *A continuous function $g : \mathbb{R} \to \mathbb{R}$ is said to be almost periodic on $\mathbb{R}$ in the sense of Bohr if for every $\varepsilon > 0$ and for every $t \in \mathbb{R}$, the set*

$$\Big\{\omega : \sup_{t\in\mathbb{R}} |g(t+\omega) - g(t)| < \varepsilon\Big\}$$

is relatively dense in $\mathbb{R}$.

Consider the set [38,52,53]

$$\mathcal{T} = \{\{t_{l_k}\} :\ t_{l_k} \in (-\infty, \infty),\ t_{l_k} < t_{l_{k+1}},\ l_k \in \mathbb{Z},\ \lim_{l_k \to \pm\infty} t_{l_k} = \pm\infty\}$$

of all unbounded and strictly increasing sequences of impulsive points of the type $\{t_{l_k}\}$ with distance $\rho\big(\{t_{l_k}^{(1)}\}, \{t_{l_k}^{(2)}\}\big)$.

For any two given $T, \bar{T} \in \mathcal{T}$ denote by $s(T \cup \bar{T}) : \mathcal{T} \to \mathcal{T}$ the map for which the set $s(T \cup \bar{T})$ forms a strictly increasing sequence.

Let $D \subset \mathbb{R}$. Denote $D_\varepsilon = \{t + \varepsilon,\ t \in D\}$, $\Theta_\varepsilon(D) = \cap\{D_\varepsilon\}$ for $\varepsilon > 0$ and $PC[J, \mathbb{R}] = \{\phi : J \to \mathbb{R} :\ \phi(t)$ is continuous everywhere except at some points $\tilde{t} \in J$ at which $\phi(\tilde{t}^-)$ and $\phi(\tilde{t}^+)$ exist and $\phi(\tilde{t}^-) = \phi(\tilde{t})\}$, where $J \subset \mathbb{R}$.

An element of the space $PC[\mathbb{R}, \mathbb{R}] \times \mathcal{T}$ will be denoted by $\Phi = (\phi(t), T)$ and the sets $(\phi_r(t), T_r) = (\phi(t + s_r), T - s_r) \subset PC[\mathbb{R}, \mathbb{R}] \times \mathcal{T}$, where $T - s_r = \{t_{l_k} - s_r,\ l_k = \pm 1, \pm 2, \dots\}$, $\{s_r\}$, $r = 1, 2, \dots$ is a sequence of real numbers, will be denoted by Φ_r.

We will also apply the following almost periodicity definitions.

Definition 5. *Consider a set of sequences*

$$\{t_{l_k}^{\mu}\},\ t_{l_k}^{\mu} = t_{l_{k+\mu}} - t_{l_k},\ l_k = \pm 1, \pm 2, \dots,\ \mu = \pm 1, \pm 2, \dots. \tag{5}$$

The set (5) is said to be uniformly almost periodic if from each infinite sequence of shifts $\{t_{l_k} - s_r\}$, $l_k = \pm 1, \pm 2, \dots, r = 1, 2, \dots,$ $s_r \in \mathbb{R}$, *we can choose a convergent subsequence in* $\mathcal{T}$.

Definition 6. *For any element* $\Phi = (\phi(t), T) \in PC[\mathbb{R}, \mathbb{R}] \times \mathcal{T}$, *a sequence* $\{\Phi_r\}$, $\Phi_r = (\phi_r(t), T_r) \in PC[\mathbb{R}, \mathbb{R}] \times \mathcal{T}$ *is said to be convergent with a limit* $\Phi = (\phi(t), T)$ *if for any* $\varepsilon > 0$, *there exists* $r_0 > 0$, *such that for* $r \geq r_0$, *we have*

$$\rho(T, T_r) < \varepsilon, \; ||\phi_r(t) - \phi(t)|| < \varepsilon$$

hold uniformly for $t \in \mathbb{R} \setminus \Theta_\varepsilon(s(T_r \cup T))$.

Definition 7. *The function* $\phi \in PC[\mathbb{R}, \mathbb{R}]$ *is said to be an almost periodic piecewise continuous function with points of discontinuity of the first kind* t_{l_k}, $\{t_{l_k}\} \in \mathcal{T}$ *if for every sequence of real numbers* $\{s'_m\}$, *there exists a subsequence* $\{s_r\}$, $s_r = s'_{m_r}$, *such that* Φ_r *is compact in* $PC[\mathbb{R}, \mathbb{R}] \times \mathcal{T}$.

We will also introduce the following assumptions:

A1. The functions a_i, $i = 1, 2, \dots, n$ are continuous on $\mathbb{R}$, almost periodic in the sense of Bohr, and there exist positive constants $\underline{a}_i$ and $\overline{a}_i$ such that $1 < \underline{a}_i \leq a_i(\chi) \leq \overline{a}_i$ for $\chi \in \mathbb{R}$.

A2. The functions $b_i(t, \chi)$ are continuous along $\chi \in \mathbb{R}$, almost periodic in the sense of Bohr along $t \in \mathbb{R}$, uniformly for $\chi \in \mathbb{R}$, and there exist almost periodic continuous functions $B_i(t) > 0$ such that

$$\frac{\overline{a}_i}{\underline{a}_i} b_i(t, \chi_1) - \frac{\underline{a}_i}{\overline{a}_i} b_i(t, \chi_2) \geq B_i(t)(\chi_1 - \chi_2),$$

for any $\chi_1, \chi_2 \in \mathbb{R}$, $\chi_1 \neq \chi_2$ and $i = 1, 2, \dots, n$.

A3. The functions f_i are continuous on $\mathbb{R}$, $f_i(0) = 0$, and there exist constants $L_i > 0$, $H_i^{(1)} > 0$ such that

$$\left| \frac{\overline{a}_i}{\underline{a}_i} f_i(\chi_1) - \frac{\underline{a}_i}{\overline{a}_i} f_i(\chi_2) \right| \leq L_i |\chi_1 - \chi_2|, \;\; |f_i(\chi)| \leq H_i^{(1)}$$

for all $\chi_1, \chi_2 \in \mathbb{R}$, $\chi_1 \neq \chi_2$, $i = 1, 2, \dots, n$.

A4. The functions c_{ij} and I_i are almost periodic in the sense of Bohr for all $i, j = 1, 2, \dots, n$.

A5. The continuous functions p_{ik} are almost periodic in the sense of Bohr and

$$p_{ik}(\chi) = \gamma_{ik} \chi, \; -2 < \gamma_{ik} < 0$$

for $\chi \in \mathbb{R}$ and all $i = 1, 2, \dots, n$, $k = \pm 1, \pm 2, \dots$.

A6. The set of sequences $\{t^\mu_{l_k}\}$, $l_k = \pm 1, \pm 2, \dots,$ $\mu = \pm 1, \pm 2, \dots$ is uniformly almost periodic, and $\inf_{l_k} t^1_{l_k} > 0$.

Remark 2. *For more details about the assumptions A1–A6, we refer to the investigations on the almost periodicity in integer-order Cohen–Grossberg and related neural network models [38,40,41,45,46,53].*

It is very well known [25,48,53] that the assumptions A1–A6 imply the existence of a subsequence $\{s_r\}$, $s_r = s'_{m_r}$ of an arbitrary sequence of real numbers $\{s'_m\}$ that "moves" Equation (2) to the following model

$$\begin{cases} {}^{C}_{t_0}D^{\nu}_{t} q_i(t) = -a_i(q_i(t)) \left[b^s_i(t, q_i(t)) - \sum_{j=1}^{n} c^s_{ij}(t) f_j(q_j(t)) - I^s_i(t) \right], \quad t \neq \theta^s_k(q), \\ q_i(t^+) = q_i(t) + \gamma_{ik} q_i(t), \; t = \theta^s_k(q), \; k = \pm 1, \pm 2, \dots. \end{cases} \tag{6}$$

Following the almost periodicity theory [52,53] for impulsive systems, we will denote the set of all systems of Equation (6) by $\mathcal{H}(2)$. For the vector representation of Equation (6), we will need the notation

$$F^s(t,q) = (F_1^s(t,q), F_2^s(t,q), \dots, F_n^s(t,q))^T,$$

where

$$F_i^s(t,q) = -a_i(q_i(t))\left[b_i^s(t,q_i(t)) - \sum_{j=1}^{n} c_{ij}^s(t) f_j(q_j(t)) - I_i^s(t)\right],$$

$i = 1,2,\dots,n$.

2.4. Lyapunov-Type Functions Definitions and Lemmas

Here, we will recall some Lyapunov functions-related definitions and lemmas [25,35]. Define the sets

$$\Omega_k = \{(t,q,\bar{q}) : \theta_{k-1}(q) < t < \theta_k(q),\ \theta_{k-1}(\bar{q}) < t < \theta_k(\bar{q}),\ q,\bar{q} \in \mathbb{R}^n\},\ k = \pm1,\pm2,\dots,$$

$$\Omega = \bigcup_{k=\pm1,\pm2,\dots} \Omega_k.$$

Definition 8. *A function $V : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$ belongs to the class V_0 if:*

1. *V is defined and continuous on Ω, V has nonnegative values, and $V(t,0,0) = 0$ for $t \in \mathbb{R}$;*
2. *V is differentiable in t and locally Lipschitz continuous with respect to its second and third arguments on each of the sets Ω_k;*
3. *For any $(t^*,q^*),(t^*,\bar{q}^*) \in \tau_k$, and each $k = \pm1,\pm2,\dots$, there exist the finite limits*

$$V(t^{*-},q^*,\bar{q}^*) = \lim_{\substack{(t,q,\bar{q})\to(t^*,q^*,\bar{q}^*)\\ (t,q,\bar{q})\in\Omega_k}} V(t,q,\bar{q}), \quad V(t^{*+},q^*,\bar{q}^*) = \lim_{\substack{(t,q,\bar{q})\to(t^*,q^*,\bar{q}^*)\\ (t,q,\bar{q})\in\Omega_{k+1}}} V(t,q,\bar{q}),$$

and $V(t^{*-},q^*,\bar{q}^*) = V(t^*,q^*,\bar{q}^*)$.

We will use the the following derivative of order ν, $0 < \nu < 1$ of a function $V \in V_0$ [25]:

$${}^C D_+^{\nu} V(t,q,\bar{q}) = \lim_{\delta\to0^+} \sup \frac{1}{\delta^{\nu}}\left[V(t,q,\bar{q}) - V(t-\delta, q-\delta^{\nu}F(t,q), \bar{q}-\delta^{\nu}F(t,\bar{q}))\right].$$

The following results follows from Lemma 1.5 in [25].

Lemma 1. *Assume that the function $V \in V_0$ satisfies for $t \geq t_0$ and $q,\bar{q} \in \mathbb{R}^n$:*

$$V(t^+, q+p_k(q), \bar{q}+p_k(\bar{q})) \leq V(t,q,\bar{q}),\ \ t = \theta_k(q),\ t = \theta_k(\bar{q}),\ k = \pm1,\pm2,\dots,$$

$${}^C D_+^{\nu} V(t,q,\bar{q}) \leq cV(t,q,\bar{q}) + d,\ t \neq \theta_k(q),\ t \neq \theta_k(\bar{q}),\ k = \pm1,\pm2,\dots,$$

where $c,d \in \mathbb{R}$.

Then, for $t \in [t_0,\infty)$, we have

$$V(t,q(t;t_0,q_0),\bar{q}(t;t_0,\bar{q}_0)) \leq V(t_0^+,q_0,\bar{q}_0)E_{\nu}(c(t-t_0)^{\nu}) + d(t-t_0)^{\nu}E_{\nu,\nu+1}(c(t-t_0)^{\nu}).$$

Remark 3. *For the case $d = 0$, we have*

$$V(t,q(t;t_0,q_0),\bar{q}(t;t_0,\bar{q}_0)) \leq V(t_0^+,q_0,\bar{q}_0)E_{\nu}(c(t-t_0)^{\nu}),\ t \geq t_0,$$

which is basically the result in Corollary 1.3 from [25].

Remark 4. *For the case $\nu = 1$ and $d = 0$, we have*

$$V(t, q(t; t_0, q_0), \bar{q}(t; t_0, \bar{q}_0)) \leq V(t_0^+, q_0, \bar{q}_0)e^{c(t-t_0)},\ t \geq t_0.$$

Remark 5. *For the continuous case (all $p_k = 0$, $k = \pm 1, \pm 2, \dots$), a similar result is presented in [7]. For fractional-order cases, more Gronwall-type differential inequalities can be found in [3,8,9,25] and the bibliography cited there.*

3. Main Almost Periodicity Results

In this section, we will state our main almost periodicity and global perfect Mittag–Leffler stability criteria for the impulsive fractional-order Cohen–Grossberg neural network model in Equation (2). These results are the first contributions to the almost periodicity theory for such fractional models and extend and generalize the results in [38,40,45,46] for integer-order models to the fractional-order case.

Theorem 1. *Suppose that assumptions A1–A6 hold and:*

(i) There exist constant $\mu_1 < 0$ such that $\mu_1(t) < \mu_1$, where $\mu_1(t)$ is the greatest eigenvalue of the symmetric matrix with entries

$$\mu_{ij}(t) = \begin{cases} -B_i(t+s_r)\underline{a}_i + |c_{ii}(t+s_l)|L_i\bar{a}_i, & i = j, \\ \dfrac{|c_{ij}(t+s_l)|L_j\bar{a}_j + |c_{ji}(t+s_l)|L_i\bar{a}_i}{2}, & i \neq j, \end{cases}$$

and $\{s_r\}$ is an arbitrary sequence of real numbers;

(ii) The impulsive functions satisfy

$$\bar{a}_i|1+\gamma_{ik}| < \underline{a}_i,\ i = 1, 2, \dots, n,\ k = \pm 1, \pm 2, \dots;$$

(iii) There exists a bounded solution $q(t, t_0, q_0)$ of (2) such that

$$||q(t, t_0, q_0)|| < \alpha,$$

where $t \geq t_0$, $\alpha > 0$.

Then, there exists a unique almost periodic solution $\beta(t)$ of the model (2) such that:

(a) $||\beta(t)|| \leq \alpha_1$, $\alpha_1 < \alpha$;

(b) $\mathcal{H}(\beta, t_{l_k}) \subset \mathcal{H}(2)$.

Proof. Consider an arbitrary sequence of real numbers $\{s_r\}$, $s_r \to \infty$ as $r \to \infty$ that moves the fractional-order neural network model in Equation (2) to a system at $\mathcal{H}(2)$, and let $t_0 \in \mathbb{R}$.

For a Lyapunov function of the type

$$V(q(t), \bar{q}(t)) = \frac{1}{2}\sum_{i=1}^{n}[v_i(q_i(t), \bar{q}_i(t))]^2,$$

where

$$v_i(q_i, \bar{q}_i) = sign(q_i - \bar{q}_i)\int_{\bar{q}_i}^{q_i}\frac{1}{a_i(\xi)}d\xi,\ i = 1, 2, \dots, n, \tag{7}$$

we have

$$\frac{1}{\bar{a}_i}|q_i - \bar{q}_i| \leq v_i(q_i, \bar{q}_i) \leq \frac{1}{\underline{a}_i}|q_i - \bar{q}_i|. \tag{8}$$

Consider the case, when $t = \theta_k(q)$, $k = \pm 1, \pm 2, \dots$. From the choice of the function $V \in V_0$, Equation (8), and A5, for $s_r, s_l \in \{s_r\}$, we obtain

$$v_i(q_i(t^+ + s_r), q_i(t^+ + s_l)) \le \left| \int_{q_i(t^+ + s_l)}^{q_i(t^+ + s_r)} \frac{1}{a_i(\xi)} d\xi \right| = \left| \int_{(1+\gamma_{ik})q_i(t+s_l)}^{(1+\gamma_{ik})q_i(t+s_r)} \frac{1}{a_i(\xi)} d\xi \right|.$$

After a substitution $\xi = \zeta(1 + \gamma_{ik})$, Equation (8) and condition (ii) of Theorem 1 imply

$$q_i(x_i(t^+ + s_r), q_i(t^+ + s_l)) \le \left| \int_{q_i(t+s_l)}^{q_i(t+s_r)} \frac{1 + \gamma_{ik}}{a_i((1 + \gamma_{ik})\zeta)} d\zeta \right|$$

$$\le \frac{\overline{a}_i}{\underline{a}_i} |1 + \gamma_{ik}| v_i(q_i(t + s_r), q_i(t + s_l)) \le v_i(q_i(t + s_r), q_i(t + s_l)).$$

Hence

$$V(q(t^+ + s_r), q(t^+ + s_l)) = \frac{1}{2} \sum_{i=1}^{n} \Big[v_i(q_i(t^+ + s_r), q_i(t^+ + s_l)) \Big]^2$$

$$\le \frac{1}{2} \sum_{i=1}^{n} [v_i(q_i(t + s_r), q_i(t + s_l))]^2 = V(q_i(t + s_r), q_i(t + s_l)). \tag{9}$$

Let $\iota \in \mathbb{R}$, and $r_0 = r_0(\iota)$ denotes the smallest r, such that $s_{r_0} + \iota \ge t_0$. From condition (iii) of Theorem 1, we have $||q(t + s_r; t_0, q_0)|| \le \alpha_1$ for $t \ge \iota$, $r \ge r_0$.

Consider a compact set C, $C \subset (\iota, \infty)$. From the choice of r_0, it follows that for any $\varepsilon > 0$, we can choose an integer $m_0(\varepsilon, \iota) \ge r_0(\iota)$ so large that for $l \ge r \ge m_0(\varepsilon, \iota)$ and $t \in \mathbb{R}$, $t \ne \theta_k(q)$, $k = \pm 1, \pm 2, \dots$, we have

$$\Big| \frac{\overline{a}_i}{\underline{a}_i} I_i(t + s_r) - \frac{\underline{a}_i}{\overline{a}_i} I_i(t + s_l) \Big| < \varepsilon \tag{10}$$

and

$$\frac{1}{2} \sum_{i=1}^{n} \Big(sign(q_i(t_0 + s_r) - q_i(t_0 + s_l)) \int_{q_i(t_0+s_l)}^{q_i(t_0+s_r)} \frac{1}{a_i(\xi)} d\xi \Big)^2 E_q(2\mu_1(t - t_0)^\nu) < \varepsilon. \tag{11}$$

Now, consider the case when $t \ge t_0$ and $(t, q(t + s_r), q(t + s_l)) \in \Omega_k$. In this case, by P2 and P3, we obtain

$$ {}_{t_0}^{C}D_t^\nu V(q(t + s_r), q(t + s_l))$$

$$\le \sum_{i=1}^{n} [v_i(q_i(t + s_r), q_i(t + s_l)) {}_{t_0}^{C}D_t^\nu v_i(q_i(t + s_r), q_i(t + s_l))]. \tag{12}$$

Furthermore, using Equation (7) and the definition of the Caputo fractional derivative of order ν, $0 < \nu < 1$, we have that

$$ {}_{t_0}^{C}D_t^\nu v_i(q_i(t + s_r), q_i(t + s_l))$$

$$= sign(q_i(t + s_r) - q_i(t + s_l)) \frac{1}{\Gamma(1 - \nu)} \int_{t_0}^{t} \Big[\int_{q_i(\sigma+s_l)}^{q_i(\sigma+s_r)} \frac{1}{a_i(\xi)} d\xi \Big]' \frac{1}{(t - \sigma)^\nu} d\sigma$$

$$= sign(q_i(t + s_r) - q_i(t + s_l)) \frac{1}{\Gamma(1 - \nu)} \int_{t_0}^{t} \Big[\frac{q_i'(t + s_r)}{a_i(q_i(t + s_r))}$$

$$- \frac{q_i'(t + s_l)}{a_i(q_i(t + s_l))} \Big] \frac{1}{(t - \sigma)^\nu} d\sigma$$

$$\le sign(q_i(t + s_r) - q_i(t + s_l)) \frac{1}{\Gamma(1 - \nu)} \int_{t_0}^{t} \Big[\frac{q_i'(t + s_r)}{\underline{a}_i} - \frac{q_i'(t + s_l)}{\overline{a}_i} \Big] \frac{1}{(t - \sigma)^\nu} d\sigma$$

$$\leq sign(q_i(t+s_r)-q_i(t+s_l))\Big[-\Big(\frac{\overline{a}_i}{\underline{a}_i}b_i(t+s_r,q_i(t+s_r))-\frac{\underline{a}_i}{\overline{a}_i}b_i(t+s_l,q_i(t+s_l))\Big)$$
$$+\sum_{j=1}^{n}\Big(\frac{\overline{a}_i}{\underline{a}_i}c_{ij}(t+s_r)f_j(q_j(t+s_r))-\frac{\underline{a}_i}{\overline{a}_i}c_{ij}(t+s_l)f_j(q_j(t+s_l))\Big)$$
$$+\frac{\overline{a}_i}{\underline{a}_i}I_i(t+s_r)-\frac{\underline{a}_i}{\overline{a}_i}I_i(t+s_l)\Big]. \quad (13)$$

Using A2 and Equation (8), we obtain

$$sign(q_i(t+s_r)-q_i(t+s_l))\Big[-\Big(\frac{\overline{a}_i}{\underline{a}_i}b_i(t+s_r,q_i(t+s_r))-\frac{\underline{a}_i}{\overline{a}_i}b_i(t+s_l,q_i(t+s_l))\Big)\Big]$$
$$=-\Big(\frac{\overline{a}_i}{\underline{a}_i}b_i(t+s_r,q_i(t+s_r))-\frac{\underline{a}_i}{\overline{a}_i}b_i(t+s_r,q_i(t+s_l)\Big)sign(q_i(t+s_r)-x_i(t+s_l))$$
$$+\frac{\underline{a}_i}{\overline{a}_i}\Big(b_i(t+s_l,q_i(t+s_l))-b_i(t+s_r,q_i(t+s_l))sign(q_i(t+s_r)-q_i(t+s_l))\Big)$$
$$\leq -B_i(t+s_r)\underline{a}_i v_i(q_i(t+s_r),q_i(t+s_l))+\frac{\underline{a}_i}{\overline{a}_i}\varepsilon. \quad (14)$$

Based on A3, A4, and Equation (8), we obtain

$$sign(q_i(t+s_r)-q_i(t+s_l))\sum_{j=1}^{n}\Big(\frac{\overline{a}_i}{\underline{a}_i}c_{ij}(t+s_r)f_j(q_j(t+s_r))-\frac{\underline{a}_i}{\overline{a}_i}c_{ij}(t+s_l)f_j(q_j(t+s_l))\Big)$$
$$\leq\sum_{j=1}^{n}\Big(\frac{\overline{a}_i}{\underline{a}_i}|c_{ij}(t+s_r)-c_{ij}(t+s_l)||f_j(q_j(t+s_r))|+|c_{ij}(t+s_l)||\frac{\overline{a}_i}{\underline{a}_i}f_j(q_j(t+s_r))-\frac{\underline{a}_i}{\overline{a}_i}f_j(q_j(t+s_l))|\Big)$$
$$\leq\varepsilon\frac{\overline{a}_i}{\underline{a}_i}\sum_{j=1}^{n}H_j^{(1)}+\sum_{j=1}^{n}|c_{ij}(t+s_l)|L_j\overline{a}_j v_j(q_j(t+s_r),q_j(t+s_l)). \quad (15)$$

Applying Equations (10)–(15), we obtain

$${}_{t_0}^{C}D_t^{\nu}V(q(t+s_r),q(t+s_l))$$
$$\leq\sum_{i=1}^{n}\Big(-B_i(t+s_r)\underline{a}_i+|c_{ii}(t+s_l)|L_i\overline{a}_i\Big)v_i^2(q_i(t+s_r),q_i(t+s_l))$$
$$+\sum_{i,j=1,\,i\neq j}^{n}|c_{ij}(t+s_l)|L_j\overline{a}_j v_i(q_i(t+s_r),q_i(t+s_l))v_j(q_j(t+s_r),q_j(t+s_l))$$
$$+\varepsilon\Big(n+\frac{\overline{a}_i}{\underline{a}_i}\Big(1+\sum_{j=1}^{n}H_j^{(1)}\Big)\Big)\sum_{i=1}^{n}v_i(q_i(t+s_r),q_i(t+s_l))$$
$$\leq 2\mu_1 V(q(t+s_r),q(t+s_l))+\varepsilon H\sqrt{V(q(t+s_r),q(t+s_l))}, \quad (16)$$

where $H=n+\frac{\overline{a}_i}{\underline{a}_i}\Big(1+\sum_{j=1}^{n}H_j^{(1)}\Big)$.

Hence, by Equation (9), Equation (16), and Lemma 1, we obtain

$$V(q(t+s_r),q(t+s_l))$$
$$\leq\Big(V(q(t_0+s_r),q(t_0+s_l))E_\nu(2\mu_1(t-t_0)^\nu)+\varepsilon H(t-t_0)^\nu E_{\nu,\nu+1}(2\mu_1(t-t_0)^\nu)^2<A\varepsilon^2,\ t\in[t_0,\infty),$$

where A is a positive constant.

Introducing $\bar{a} = \max_{1\leq i\leq n} \bar{a}_i$, the above estimate leads to

$$||(q(t+s_r) - q(t+s_l)|| \leq \bar{a}V(q(t+s_r), q(t+s_l)) < \bar{a}A\varepsilon^2,$$

which implies the existence of a function $\beta(t) = (\beta_1(t), \beta_2(t), \ldots, \beta_n(t))^T$ such that $q(t+s_r) - \beta(t) \to 0$ as $r \to \infty$. From the arbitrariness of ι, it follows that $\beta(t)$ is defined uniformly on $t \in \mathbb{R}$.

The proof of the fact that $\lim_{r\to\infty} {}_{t_0}^{C}D_t^{\nu} q(t+s_r)$ exists uniformly on all compact subsets of $\mathbb{R}$ is similar.

Since $\lim_{r\to\infty} {}_{t_0}^{C}D_t^{\nu} q(t+s_r) = {}_{t_0}^{C}D_t^{\nu}\beta(t)$ and $\lim_{r\to\infty} \theta_k^s(q) = \lim_{r\to\infty} \theta_k(q(t+s_r)) = \theta_k(q)$, we have for $t \neq \theta_k^s(q)$,

$$\begin{aligned} {}_{t_0}^{C}D_t^{\nu}\beta(t) &= \lim_{r\to\infty} \Big(F^s(t+s_r, q(t+s_r)) - F^s(t+s_r, \beta(t)) + F^s(t+s_r, \beta(t))\Big) \\ &= F(t, \beta(t)),\ t \neq \theta_k(q). \end{aligned} \tag{17}$$

and for $t = \theta_k^s(q)$,

$$\begin{aligned} \beta(t^+) - \beta(t^-) &= \lim_{r\to\infty}(q(t+s_r+0) - q(t+s_r-0)) \\ &= \lim_{r\to\infty} p_k^s(q(t+s_r)) = p_k(\beta(t)),\ t = \theta_k(q),\ k = \pm1, \pm2, \ldots, \end{aligned} \tag{18}$$

which imply that the function $\beta(t)$ is a solution of Equation (2).

Finally, in order to prove the almost periodicity of the solution $\beta(t)$, we will use again the sequence $\{s_r\}$ that moves Equation (2) to $\mathcal{H}(2)$ and the function

$$V(\beta(\sigma), \beta(\sigma+s_r-s_l)) = \frac{1}{2}\sum_{i=1}^{n}[v_i(\beta(\sigma), \beta(\sigma+s_r-s_l))]^2.$$

Applying similar arguments as above, we obtain

$$V((\beta(t+s_r), \beta(t+s_l)) < A\varepsilon^2$$

and, hence, for $l \geq r \geq r_0(\varepsilon)$, we obtain

$$||\beta(t+s_r) - \beta(t+s_l)|| < \bar{a}A\varepsilon^2, \tag{19}$$

$$\rho(t_{l_k} + s_r, t_{l_k} + s_l) < \varepsilon. \tag{20}$$

Therefore, Equations (19) and (20) imply that the sequence of functions $\beta(t+s_r)$ converges uniformly to the solution $\beta(t)$ as $r \to \infty$.

The properties (a) and (b) follow directly.

The proof of Theorem 1 is completed. □

One of the main qualitative properties of the solutions of neural network models, including almost periodic solutions, is their stability. That is why there exists numerous stability results for different types of solutions of integer-order Cohen–Grossberg neural networks [21–24,39,42,43]. The most important stability concept for neural network models is that of the exponential stability [46]. For fractional-order systems, the corresponding stability notion is that of Mittag–Leffler stability introduced in [69] (see also [19,25,28–31]). By the next definition, we will introduce the notion of global perfect Mittag–Leffler stability for a solution of the fractional neural network model (Equation (2)).

Definition 9. *A solution $q(t)$ of Equation (2) with an initial value $q_0 \in \mathbb{R}^n$ is:*

(a) equi-bounded if

$$(\forall Q > 0)(\forall t_0 \in \mathbb{R})(\exists R > 0)(\forall q_0 \in \mathbb{R}^n :\ ||q_0|| < Q)(\forall t \geq t_0) :\ ||q(t)|| < R.$$

(b) globally Mittag–Leffler stable if for any $q_0^1 \in \mathbb{R}^n$ such that $||q_0 - q_0^1|| < R$ $(R > 0)$ and $t_0 \in \mathbb{R}$, there exists a constant $\Upsilon > 0$ such that

$$||q(t,t_0,q_0) - q^1(t,t_0,q_0^1)|| \leq \{m(q_0 - q_0^1)E_\nu(-\Upsilon(t-t_0)^\nu),\ t \geq t_0,$$

where $m(0) = 0$, $m(q) \geq 0$, and $m(q)$ is Lipschitz with respect to $q \in \mathbb{R}^n$, $||q|| < Q$.

(c) globally perfectly Mittag–Leffler stable if it is globally Mittag–Leffler stable and the number R in a) is independent of $t_0 \in \mathbb{R}$.

Theorem 2. *Suppose that conditions of Theorem 1 hold. Then, the almost periodic solution $\beta(t)$ of Equation (2) is globally perfectly Mittag–Leffler stable.*

Proof. Let $\beta(t)$ be the almost periodic solution of Equation (2), and $\beta^1(t)$ be an arbitrary solution of Equation (6).

Denote

$$\hat{\beta}(t) = \beta^1(t) - \beta(t),$$
$$F^s(t,\hat{\beta}(t)) = F^s(t,\hat{\beta}(t)+\beta(t)) - F^s(t,\beta(t))$$

and consider the model

$$\begin{cases} {}_{t_0}^{C}D_t^{\nu}\hat{\beta}(t) = F^s(t,\hat{\beta}(t)),\ t \neq \theta_k^s(\hat{\beta}), \\ \hat{\beta}(t^+) = \hat{\beta}(t) + \gamma_{ik}\hat{\beta}(t),\ t = \theta_k^s(\hat{\beta}),\ k = \pm 1, \pm 2, ... \end{cases} \tag{21}$$

If we take the Lyapunov function to be $W(t,\hat{\beta}) = V(t,\beta,\beta+\hat{\beta})$, then apply Lemma 1, we conclude that the zero solution $\hat{\beta}(t) = 0$ of Equation (21) is globally perfectly Mittag–Leffler stable. Hence, the solution $\beta(t)$ of Equation (2) is globally perfectly Mittag–Leffler stable. This proves Theorem 2. □

The last part of our results related to the almost periodicity properties of Equation (2) is to study the effects of uncertain parameters on such properties. To this end, we consider uncertain parameters $\tilde{a}_i, \tilde{c}_{ij}, \tilde{f}_j, \tilde{I}_i \in C[\mathbb{R},\mathbb{R}]$, $\tilde{b}_i \in C[\mathbb{R}^2,\mathbb{R}_+]$ $\tilde{\gamma}_{ik} \in \mathbb{R}$, $i,j = 1,\ldots,n$, $k = \pm 1, \pm 2, \ldots$, and the following uncertain fractional order Cohen–Grossberg neural network model with variable impulsive perturbations corresponding to Equation (2)

$$\begin{cases} {}_{t_0}^{C}D_t^{\nu} q_i(t) = -(a_i(q_i(t)) + \tilde{a}_i(q_i(t)))\Big[(b_i(t,q_i(t)) + \tilde{b}_i(t,q_i(t))) \\ \qquad - \sum_{j=1}^{n}(c_{ij}(t)+\tilde{c}_{ij}(t))(f_j(q_j(t)) + \tilde{f}_j(q_j(t))) \\ \qquad - I_i(t) - \tilde{I}_i(t)\Big],\quad t \neq \theta_k(q), \\ q_i(t^+) = q_i(t) + \gamma_{ik}q_i(t) + \tilde{\gamma}_{ik}q_i(t),\ t = \theta_k(q),\ \ k = \pm 1, \pm 2, \ldots, \end{cases} \tag{22}$$

where $i = 1,2,\ldots,n$, $t \in \mathbb{R}$.

Definition 10. *The almost periodic solution $\beta(t)$ of Equation (2) is globally, perfectly, and robustly Mittag–Leffler stable if for $t \in \mathbb{R}$, $q_0 \in \mathbb{R}^n$, and for any $\tilde{a}_i, \tilde{b}_i, \tilde{c}_{ij}, \tilde{f}_j, \tilde{I}_i, \tilde{\gamma}_{ik}$, $i,j = 1,\ldots,n$, $k = \pm 1, \pm 2, \ldots$, the almost periodic solutions of Equation (22) are globally perfectly Mittag–Leffler stable.*

For the global perfect robust Mittag–Leffler stability of the almost periodic solution, we will need the following assumptions:

A7. The functions $\tilde{a}_i$ are such that

$$\tilde{a}_i^+(\chi) \in [\underline{a}_i - a_i, \overline{a}_i - a_i],$$

where $\tilde{a}_i^+ = \sup_{\chi \in \mathbb{R}} \tilde{a}_i(\chi)$, $i = 1, 2, \ldots, n$.

A8. The functions $\tilde{b}_i(t, \chi)$ are continuous along $\chi \in \mathbb{R}$, almost periodic in the sense of Bohr along $t \in \mathbb{R}$, uniformly for $\chi \in \mathbb{R}$, and there exist an almost periodic continuous functions $\tilde{B}_i(t) > 0$ such that

$$\frac{\overline{a}_i}{\underline{a}_i}(b_i(t,\chi_1) + \tilde{b}_i(t,\chi_1)) - \frac{\underline{a}_i}{\overline{a}_i}(b_i(t,\chi_2) + \tilde{b}_i(t,\chi_2)) \geq \tilde{B}_i(t)(\chi_1 - \chi_2),$$

for any $\chi_1, \chi_2 \in \mathbb{R}$, $\chi_1 \neq \chi_2$ and $i = 1, 2, \ldots, n$.

A9. The functions $\tilde{c}_{ij}(t)$, $\tilde{I}_i(t)$ are almost periodic in the sense of Bohr for $i, j = 1, 2, \ldots, n$.

A10. There exist constants $\tilde{L}_i > 0$, $\tilde{H}_i^{(1)} > 0$ such that

$$\left|\frac{\overline{a}_i}{\underline{a}_i}\tilde{f}_i(\chi_1) - \frac{\underline{a}_i}{\overline{a}_i}\tilde{f}_i(\chi_2)\right| \leq \tilde{L}_i|\chi_1 - \chi_2|, \ \tilde{f}_i(\chi)| \leq \tilde{H}_i^{(1)}$$

and $\tilde{f}_i(0) = 0$, for all $\chi_1, \chi_2 \in \mathbb{R}$, $\chi_1 \neq \chi_2$, $i = 1, 2, \ldots, n$.

A11. The unknown constants $\tilde{\gamma}_{ik}$ are such that $\tilde{\gamma}_{ik} \in [-\gamma_{ik} - 2, -\gamma_{ik}]$ and

$$\overline{a}_i|1 + \gamma_{ik} + \tilde{\gamma}_{ik}| < \underline{a}_i, \ i = 1, 2, \ldots, n, \ k = \pm 1, \pm 2, \ldots.$$

The proof of the next result is similar to the proof of Theorem 1.

Theorem 3. *Assume that:*

1. *Conditions of Theorem 1 are satisfied;*
2. *Assumptions A7–A11 hold;*
3. $\tilde{\mu}_1(t) < 0$, *where* $\tilde{\mu}_1(t)$ *is the greatest eigenvalue of the matrix*

$$\tilde{\mu}_{ij}(t) = \begin{cases} -\tilde{B}_i(t+s_r)\underline{a}_i + |c_{ii}(t+s_l) + \tilde{c}_{ii}(t+s_l)|(L_i + \tilde{L}_i)\overline{a}_i, & i = j, \\ \dfrac{|c_{ij}(t+s_l)|(L_j + \tilde{L}_j)\overline{a}_j + |c_{ji}(t+s_l)|(L_i + \tilde{L}_i)\overline{a}_i}{2}, & i \neq j. \end{cases}$$

Then, the almost periodic solution of the fractional impulsive Cohen–Grossberg neural network in Equation (2) is globally, perfectly, and robustly Mittag–Leffler stable.

Proof. It follows from the condition 1 of Theorem 3, there exists a globally perfectly Mittag–Leffler stable almost periodic solution $\beta(t)$ of Equation (2). The proof of its global perfect robust Mittag–Leffler stability follows directly from Definition 10. Thus, the proof of Theorem 3 is completed. □

4. Examples

Example 1. *In order to demonstrate our almost periodicity results, we consider Equation (2) for* $n = 2$, *or we consider the following fractional impulsive Cohen–Grossberg neural network model*

$$\begin{cases} {}_{t_0}^{C}D_t^{\nu} q_i(t) = -a_i(q_i(t))\left[b_i(t, q_i(t)) - \sum_{j=1}^{2} c_{ij}(t) f_j(q_j(t)) - I_i(t)\right], \quad t \neq \theta_k(q), \\ q_i(t^+) = q_i(t) + \gamma_{ik} q_i(t), \ t = \theta_k(q), \ k = \pm 1, \pm 2, \ldots, \end{cases} \tag{23}$$

where $i = 1, 2$, $q(t) = (q_1(t), q_2(t))^T$, $t \in \mathbb{R}$, $f_i(q_i) = \frac{1}{2}\left(\frac{\underline{a}_i}{\overline{a}_i}|q_i + 1| - \frac{\overline{a}_i}{\underline{a}_i}|q_i - 1|\right)$, $a_1(q_1) = 1 - 0.3\sin(q_1\sqrt{3})$, $a_2(q_2) = 0.2 + 0.4\sin(q_2\sqrt{3})$, $b_i(t, \chi) = \frac{\overline{a}_i}{\underline{a}_i}(5 + \sin(t\sqrt{3}))\chi$, $i = 1, 2$, $\chi \in \mathbb{R}$, $I_1(t) = \sin t$, $I_2(t) = \cos t$,

$$(c_{ij}(t)) = \begin{pmatrix} c_{11}(t) & c_{12}(t) \\ c_{21}(t) & c_{22}(t) \end{pmatrix} = \begin{pmatrix} -0.7 + 0.1\sin(t\sqrt{2}) & 0.4 - 0.2\cos(t\sqrt{2}) \\ 0.5 - 0.1\cos(t\sqrt{2}) - 0.2\cos t & -0.2 - 0.4\sin(t\sqrt{2}) \end{pmatrix},$$

$\theta_k(q) = ||q|| + k$, $k = \pm 1, \pm 2, \ldots$.

From the choice of the system parameters for $t \neq \theta_k(q)$, $k = \pm 1, \pm 2, \ldots$ *we conclude that all assumptions A1–A4 are satisfied for*

$$\underline{a}_1 = 0.7, \quad \overline{a}_1 = 1.3, \quad \underline{a}_2 = 0.2, \quad \overline{a}_2 = 0.6,$$
$$B_i(t) = 5 + \sin(t\sqrt{3}), \quad L_1 = L_2 = 1,$$

and the greatest eigenvalue of the symmetric matrix

$$(\mu_{ij}(t)) = \begin{pmatrix} \mu_{11}(t) & \mu_{12}(t) \\ \mu_{21}(t) & \mu_{22}(t) \end{pmatrix}$$

is negative.

In addition, we assume that the impulsive functions $p_{ik}(q_i) = \gamma_{ik}q_i$ *at* $t = \theta_k(q)$ *are such that*

$$|1 + \gamma_{1k}| < \frac{0.7}{1.3} < 1, \; |1 + \gamma_{2k}| < \frac{0.2}{0.6} < 1 \tag{24}$$

for any $k = \pm 1, \pm 2, \ldots$ *and assumption A6 is satisfied.*

Thus, all conditions of Theorem 1 are satisfied. Therefore, there exists a unique almost periodic solution $\beta(t)$ *of Equation (23). Furthermore, since the conditions of Theorem 2 hold, the almost periodic solution is globally perfectly Mittag–Leffler stable.*

Example 2. *Consider the impulsive fractional neural network model of Cohen–Grossberg-type in Equation (23) with uncertain parameters* $\tilde{a}_i, \tilde{b}_i, \tilde{c}_{ij}, \tilde{f}_j, \tilde{I}_i, \tilde{\gamma}_{ik}$, $i, j = 1, 2$, $k = \pm 1, \pm 2, \ldots$

$$\begin{cases} {}_{t_0}^{C}D_t^{\nu} q_i(t) = -(a_i(q_i(t)) + \tilde{a}_i(q_i(t)))\Big[(b_i(t, q_i(t)) + \tilde{b}_i(t, q_i(t))) \\ \qquad - \sum_{j=1}^{2}(c_{ij}(t) + \tilde{c}_{ij}(t))(f_j(q_j(t)) + \tilde{f}_j(q_j(t))) \\ \qquad - I_i(t) - \tilde{I}_i(t)\Big], \quad t \neq \theta_k(q), \\ q_i(t^+) = q_i(t) + \gamma_{ik}q_i(t) + \tilde{\gamma}_{ik}q_i(t), \; t = \theta_k(q), \;\; k = \pm 1, \pm 2, \ldots, \end{cases} \tag{25}$$

where $t \in \mathbb{R}$.

If the uncertain parameters satisfy conditions 2 and 3 of Theorem 3, then the almost periodic solution of Equation (23) is globally, perfectly, and robustly Mittag–Leffler stable. We can also mention, that if, for example, condition A11 is not satisfied, we can not make any conclusion about the global perfect robust Mittag–Leffler stability behavior of the almost periodic state.

Remark 6. *As we can see from the presented examples, the proposed impulsive control technique may be efficiently applied in the global perfect Mittag–Leffler stability analysis of the almost periodic solutions.*

5. Conclusions

In this paper, we investigate an impulsive fractional-order Cohen–Grossberg neural network model. The notion of almost periodicity is extended to the model under consideration. Using Lyapunov-type functions, criteria for the existence and uniqueness of almost periodic waves for the proposed fractional-order neural network model are established. The concept of global perfect Mittag–Leffler stability of the almost periodic solution is also introduced and studied. In addition, the almost periodicity of the model under uncertain parameters is investigated. Our qualitative criteria generalize and complement some existing almost periodicity results for fractional Cohen–Grossberg neural network models to the impulsive case. We propose an impulsive control technique via variable impulsive perturbations. The proposed technique can be applied in the investigation of qualitative properties of different types of fractional-order impulsive neural network models.

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