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Review

A Review of Research on Mathematics Teacher Educator Knowledge: Mapping the Terrain

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Abstract: Over the past two decades, the landscape of research on mathematics teacher educators (MTEs) has grown considerably. One particular area of interest has focused on MTE knowledge and the ways in which it is developed and used in teaching practice. However, studies have conceptualized MTE knowledge in different ways and have employed considerably different methodologies and approaches to its study. In an effort to understand this varied landscape, we conducted an extensive review of research on the nature and development of MTE knowledge. This review provides a broad descriptive analysis of the existing theoretical and empirical research on MTEs' knowledge, explores the theoretical underpinnings of the existing frameworks for and studies on MTE knowledge, and considers implications for future research.

Keywords: mathematics teacher educator; teacher educator knowledge; mathematical knowledge for teaching teachers; mathematics education

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1. Introduction

The mathematics education community has long been concerned with ensuring the adequate preparation of prospective teachers in learning to teach mathematics effectively [1,2]. More recently, the issues of what mathematics teachers need to know [3–5] and how to prepare prospective teachers (PTs) as part of teacher preparation programs (e.g., [6–8]) have become central foci in both research and policy arenas in many countries. Despite such attention to teacher knowledge and preparation, research on mathematics teacher educators (MTEs) is still in its infancy. The term "MTEs" refers to individuals who work with PTs in a variety of contexts to develop and improve the teaching of mathematics [9], including mathematicians, graduate students, mathematics education researchers, and classroom teachers. As such, these individuals play a critical role in the mathematical preparation of PTs.

Over the past two decades, the landscape of research on MTEs has grown considerably. This growth is reflected by the increase in the number of working groups at international conferences [10–12], special issues of journals [13,14], and international handbooks [1,11,15], all of which have taken up questions focused on MTE knowledge. However, researchers have studied MTEs and their knowledge in a variety of ways, employing considerably different methodologies, approaches, and theoretical lenses. For example, some researchers have conducted self-studies of their own practice to highlight the nature of their knowledge used when teaching teachers (e.g., [16,17]). Other researchers who have studied MTE knowledge have used research on teacher knowledge to conceptualize what MTE knowledge entails in different ways (e.g., [18,19]). Still, others have conducted research on MTE knowledge through the lens of MTEs' instructional goals and design decisions to understand the knowledge they draw on when teaching teachers (e.g., [20,21]. Despite the growing number of studies focused on MTEs, to date, there has been no comprehensive review of research focused on MTE knowledge.

In an effort to understand this varied landscape, the authors conducted an extensive review of research on the nature and development of MTE knowledge. This review provides a broad descriptive analysis of the existing theoretical and empirical research on MTEs' knowledge, explores the theoretical underpinnings of the existing frameworks for MTE knowledge, and considers implications for future research. We focused our review primarily on research related to university-based MTEs who teach courses for elementary PTs (we define elementary to include students aged 5–12 years old). We chose university-based MTEs because of the recent calls by various organizations and governmental education authorities for reforming teacher preparation programs to incorporate particular content standards and core practices and thus increase the accountability of these programs (e.g., [3,5,22]). Notably absent from all of these recommendations is guidance for MTEs in implementing their suggestions and for the preparation and professional support of MTEs who do this work. The quality of teacher preparation programs largely depends on the competence and expertise of its teacher educators [7,23], yet little is known about the qualifications of the faculty and staff who teach such programs (e.g., [24]). Moreover, much of the extant research on MTEs draws on the construct of mathematical knowledge for teaching (MKT; [4]), a model of teacher knowledge that is grounded in elementary school teaching. Thus, research on MTEs is largely focused on MTEs who teach elementary PTs.

In this review, we focus on the following questions: (1) What differences and similarities exist in how MTE knowledge has been conceptualized? (2) How has MTE knowledge been studied? and (3) What are the different contexts in which MTE knowledge is developed?

The currently changing context surrounding teacher preparation programs provides a critical opportunity for the field to seriously consider how MTEs who work with elementary PTs are prepared and supported. This population includes a diverse set of individuals who come to this work with varied backgrounds and are often not professionally prepared [24]. By mapping the terrain of research on MTE knowledge, our goals are to advance the field's collective understanding of MTE knowledge and identify pathways for future research on MTEs that will be generative and build on each other in systematic ways.

2. Conceptualizing MTE Knowledge

To understand the nature of the extant research on MTE knowledge, it is necessary to consider what constitutes teacher knowledge. Broadly defined, knowledge is the information and skills that teachers develop through experience and education. Researchers have long recognized the tension that exists for teachers between their general content-specific knowledge, developed in teacher education programs, and the craft knowledge that is developed through teaching practice [25,26]. Craft knowledge, for example, includes "know-how for teaching based on past experiences, empirical data, and well-reasoned arguments and predictions" (Hiebert & Morris p. 476 [27]), knowledge that is developed through experience of working with students. Like others [28,29], we argue that teacher knowledge is intimately related to teaching practice—teachers' knowledge is further developed and enhanced over time as they teach students and interact with their colleagues.

In their review of research on knowledge and teacher learning, Cochran-Smith and Lytle explicated three prominent perspectives for how teacher knowledge is developed [29]. The perspective most closely aligned with this review is knowledge of practice. Notably, the other two perspectives (i.e., knowledge in practice and knowledge for practice) in their review involve a more formal knowledge base for teachers to learn, one that includes both theoretical and practical elements. For MTEs, no such formal, agreed-upon knowledge base currently exists—a point we return to later in the discussion.

Cochran-Smith and Lytle describe knowledge of practice as knowledge derived from inquiry where teachers use their own or others' classrooms as sites for investigation in order to problematize and actively investigate their own or others' teaching practices [29]. From this perspective, the knowledge teachers need to be effective is developed as they draw on resources, such as research studies or the knowledge of colleagues, to inform and further develop their practice over time. Such knowledge can be derived from reflective self-studies or inquiry communities where teachers examine their own practice, decision making, and instructional goals and the ways these aspects of practice influence their instructional interactions and ultimately student learning. Hiebert and Morris describe such knowledge as that which both new and more experienced teachers access and draw upon to support students in achieving specific goals and outcomes [27]. Underlying this perspective is an assumption that teachers, both new and experienced, develop their knowledge over time by problematizing and inquiring into their own and others' practices. As such, a knowledge-of-practice perspective elevates the development of practical or craft knowledge. We extend these definitions to include MTEs as the teachers and PTs as the students.

Implicit in a knowledge-of-practice perspective is a sociocultural perspective on how knowledge is developed through a person's interactions and experiences with the physical, social, and cultural contexts in which they participate [30,31]. MTEs develop and refine their knowledge not only through their interactions with colleagues as part of formal or informal professional learning experiences, but also through their interactions with the PTs they teach—interactions that are influenced by the broader social and cultural contexts in which MTEs' teaching practice occurs. In this review, we are concerned with the nature of MTE knowledge and the ways in which it is developed through teaching practice.

Also central to this review is an assumption that the knowledge needed by MTEs for their work with teachers differs from the knowledge needed by teachers for working with students. Researchers generally agree that the work of MTEs involves working with PTs and/or practicing teachers to develop their knowledge of teaching practice [9]. As there is considerable diversity in the nature of MTEs' work, the range of expertise shared by MTEs is similarly diverse, involving varying levels of mathematical expertise, pedagogical expertise, and/or expertise derived from their experiences as teachers [32].

While there are similarities between the knowledge needed by MTEs and teachers, there are also important differences. Mason (1998) argues that teachers need to be able to engineer instructional situations in which students experience a shift in their attention to become aware of ideas and concepts of which they were previously unaware. However, in addition to developing new content knowledge, MTEs need to support PTs in shifting their awareness to understand how the concepts they are learning connect to their future teaching of students. This requires MTEs to also understand what is involved in teaching mathematics to students. To simultaneously enhance PTs' own content learning and connect that content learning to teaching students—work that is different from what teachers do with students—MTEs need mathematical knowledge for teaching teachers (MKTT). According to Zopf, such knowledge is different from teacher knowledge in three significant ways: (1) students approach learning mathematics with only informal understanding, whereas PTs bring with them more formal, albeit often limited and procedural, knowledge; (2) teachers teach mathematics while MTEs teach MKT; and (3) the purposes of teaching students and the purposes of teaching PTs are different. While students often learn mathematics to participate in school and society writ large, PTs learn MKT to teach mathematics to students [33]. Thus, like Beswick and Chapman [10], we consider MTE knowledge to be an elaborated extension of teacher knowledge that also includes subdomains of MTE knowledge that are characteristically different from teacher knowledge.

3. Method

Literature Search

We define research related to MTEs as theoretical and empirical peer-reviewed studies and dissertations. Recognizing that there are many definitions of research, we use Creswell's idea that research constitutes a process in which information is collected and analyzed to inform one's understanding of a topic or phenomenon [34].

We conducted a literature search using ERIC and dissertation abstracts using the following search terms: (1) *mathematics teacher educator(s)* AND *knowledge*, (2) *mathematics* AND *teacher educator* AND *knowledge*, (3) *teacher trainer* AND *knowledge*, and (4) *mathematics* AND *teacher trainer*. We also searched edited books and book series that were cited in many of the articles we reviewed, recent special issues of peer-reviewed journals, and conference proceedings of the Research in Undergraduate Mathematics Education (RUME) annual meetings, as these were not captured in the ERIC search results. The search covered articles written in English and published in peer-reviewed scientific journals or conference proceedings, as well as dissertations. Hereafter, we use the term "articles" to refer to the resulting set of scientific studies, books, book chapters, dissertations, and conference proceedings. Because Shulman's seminal article about teacher knowledge was published in 1986 [26] and the majority of the research on MTEs emerged in the early 2000s, we confined the search to the years 1986–2021.

4. Selection of Articles

The literature search resulted in 755 articles (including 104 dissertations) (see Figure 1). Two steps were taken to identify articles relevant for our review. First, the authors evenly divided up and reviewed the 755 abstracts, selecting those that (1) focused on MTEs working with PTs in institutions of higher education and (2) the research phenomenon studied included MTE knowledge or its development (beyond just including implications for MTE knowledge). In the case that the assigned author could not easily determine whether to include an article or not, which occurred in approximately one-third of the reviews, these abstracts were then assigned to a different author for an additional review. During this process, any inconsistencies between authors were discussed at weekly meetings. Selections were then discussed as a whole group until a consensus was reached. A significant portion of articles did not meet our criteria because they did not explicitly focus on MTE knowledge. For example, the majority of articles focused on developing tasks and lessons for, or collaborations around, designing courses for PTs, where MTEs were only mentioned in the implications of the studies. This first step resulted in 87 remaining articles (including 6 dissertations).

Second, three authors divided up these remaining articles and individually reviewed each one in its entirety, summarizing each in terms of research goals, method(s), research questions, researcher roles, and findings. As we read and organized the articles, we inductively developed and refined a set of categories. Specifically, we categorized articles as empirical or theoretical. We then described the relationship between the key ideas or research questions in the article to MTE knowledge. For example, we considered whether the research questions were about MTE knowledge or knowledge development and grouped articles accordingly. Then, we identified the main argument or focus of the article (e.g., illustration of an analytic approach or description of a course or workshop). Notably, the purpose or focus of the research was not always explicitly stated in the same terms. In those instances, the authors inferred a research purpose based on the framework and/or discussion sections in the articles. This process led to the three main categories in Table 1, which includes a more detailed description of these summarization categories. This work was followed by group discussions until a consensus was reached. As a result, 50 articles were excluded based on this process. This second step resulted in 37 articles (see Figure 1).

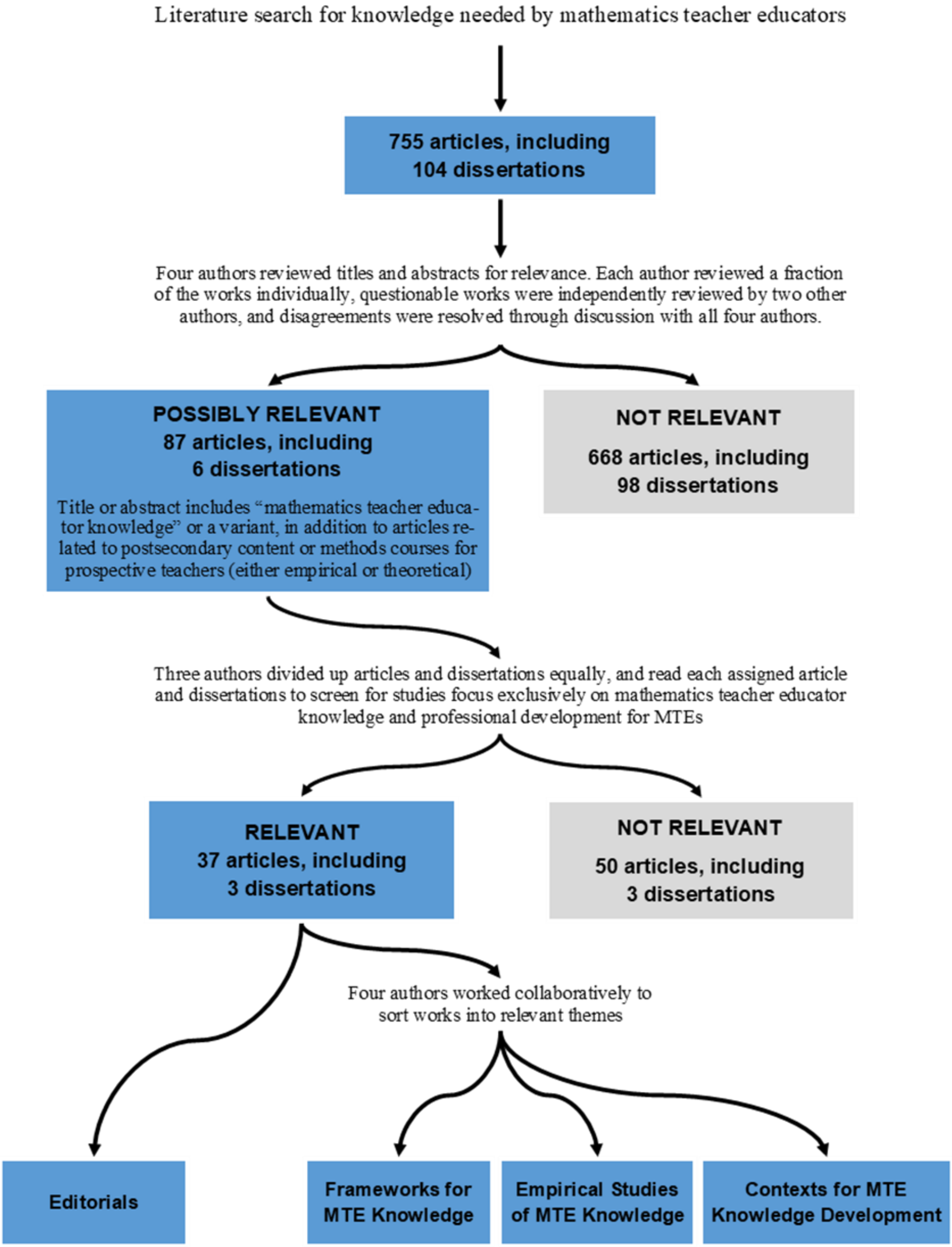

Figure 1. Outline of inclusion/exclusion process and numbers of works at each stage.

Table 1. Overview of categories of reviewed articles (n = 33 *).

Category	Subcategory	Description
Frameworks for MTE Knowledge (n = 14)	Complete Frameworks for MTE Knowledge (n = 9)	Proposes a complete framework for MTE knowledge; most articles are theoretical
	Components of a Framework for MTE Knowledge (n = 5)	Proposes different components of a framework for MTE knowledge
Empirical Studies of MTE Knowledge (n = 10)	Studies of MTE Knowledge (n = 6)	Focuses on the knowledge MTEs use when teaching PTs
	Studies of Relationship Between MTE Knowledge and Practice (n = 4)	Describes different aspects of MTE instructional practice and decision making
Contexts for MTE Knowledge Development (n = 9)	Conducting Research to Develop MTE Knowledge (n = 2)	Describes how MTEs conducting research in math teacher education enhances MTE knowledge
	Professional Learning to Develop MTE Knowledge (n = 6)	Describes PL workshops, experiences, and coursework for MTEs designed to further develop their knowledge
	Developing MTE Knowledge Through Self-Study (n = 1)	Focuses on MTEs' conducting self-studies of their own teaching practice and describes it as a means by which MTEs can further develop their knowledge

Note: * Shaughnessy et al. [35]. was included in two categories.

5. Review of Articles

The 37 remaining articles formed the basis of our review. The results are organized into three main categories, as described in Table 1: Frameworks for MTE Knowledge, Empirical Studies of MTE Knowledge, and Contexts for MTE Knowledge Development. Like Selmer and colleagues (2016) [36], we recognize that researchers use various terms, often interchangeably, to mean both similar and different things (e.g., domain and elements). For clarity, we use *MTE knowledge* to refer to the larger domain of knowledge MTEs require to teach elementary PTs and the terms *subdomains* and *components* of MTE knowledge to refer to the types of knowledge of varying grain sizes within the overall domain of MTE knowledge.

Finally, our review identified articles (n = 5) relevant to MTE knowledge, including two editorials in special issues of journals [14,30], two introductory chapters to edited volumes of research on and by MTEs [9,11], and a brief report of a discussion group at a professional conference [10]. We highlight these articles separately at the beginning of the results section as they discuss the state of research on MTE knowledge at the time they were published and thus serve a different purpose than the articles included in Table 1.

6. Results

Summary of the State of Research on MTE Knowledge

In an introductory chapter to an edited volume focused on the professional learning of MTEs, Jaworski described the similarities and differences between teacher knowledge and MTE knowledge, demonstrating the unique nature of the latter [9]. She argued that one key difference in these knowledge bases is that, unlike teachers, MTEs need to understand research on teachers and the broader systems in which teachers work (e.g., classrooms, schools, and communities), as well as knowledge of the professional and research literature on mathematics teaching and learning. She concluded by describing the evolution of research on teachers, from largely cognitive and psychological to sociocultural perspectives on teachers and teacher learning, as well as the implications of this trajectory for research on MTEs' knowledge. Four years later, in a discussion group led by Beswick and Chapman at the 12th International Congress on Mathematical Education, participants discussed how

MTE knowledge is acquired, who conducts research on MTEs (i.e., self-studies and others), and the applicability of teacher knowledge types to MTEs [10]. While the report on this discussion is quite brief, it reflects several strands of research that were emerging at the time (e.g., knowledge required by MTEs). Later, Jaworski and Huang [30] and Beswick and Goos [14] both wrote editorials to special issues of journals describing the broader state of research on MTEs. Both sets of authors examined a range of extant studies on MTE knowledge and posed questions and directions for future research. A common theme in the editorials, published four years apart, is the use of different teacher knowledge frameworks as foundations for conceptualizing MTE knowledge.

Most recently, in their introductory chapter of an edited volume focused on the learning and development of MTEs, Goos and Beswick highlighted the three themes reflected in the book: the nature of MTE expertise; the learning and development of MTEs; and the methodological challenges in researching MTE expertise, learning, and development [11]. Relevant to this review, Goos and Beswick described various frameworks for MTE knowledge, many of which are included in this review, identifying similarities across the frameworks that point to the specialized nature of MTE knowledge. They also highlighted past research efforts to apply teacher knowledge frameworks to MTEs and identify ways in which teacher knowledge differs from MTE knowledge—both themes being reflected in the articles we reviewed. Goos and Beswick argued that MTEs require a kind of "meta-knowledge" that includes the knowledge needed by math teachers as well as knowledge that is unique to MTEs. Collectively, these articles outline the evolution of research on MTE knowledge across two decades.

7. Frameworks for MTE Knowledge

As a result of our review, we found 14 articles that proposed either comprehensive frameworks or parts of frameworks for conceptualizing MTE knowledge (Table 2). In the following sections, we discuss the differences in both the structures and conceptualizations of MTE knowledge as reflected in these articles.

Table 2. Frameworks for MTE knowledge (n = 14).

Author(s)	Year	Origin	Research Purpose
Comprehensive Frameworks for MTE Knowledge (n = 9)			
Castro Superfine, Prasad, Welder, Olanoff, and Eubanks-Turner	2020	USA	Proposes a conceptualization of MKTT that extends from and contains analogous domains of MKT [4]
Chavout	2009	USA	Proposes a conceptualization of MTE knowledge that builds on the MKT framework [4]; suggests an "MTE as researcher" knowledge subdomain
Escudero-Avila, Montes, and Contreres	2021	Mexico and Spain	Proposes a conceptualization of MTE knowledge with seven subdomains based on a review of research with and about mathematics teachers
Hauk, Jackson, and Tsay	2017	USA	Builds on Ball et al.'s MKT framework [4] to propose a framework for the "mathematical knowledge for teaching future teachers"
Mason	1998	UK	Conceptualizes MTE knowledge as including different levels of awareness
Olanoff, Welder, Prasad, and Castro Superfine	2018	USA	Proposes a conceptualization of MKTT that extends from and contains analogous domains of MKT [4]
Perks and Prestage	2008	UK	Proposes an MTE knowledge "tetrahedron" based on their teacher knowledge "tetrahedron" [37]
Shaughnessy, Garcia, Selling, and Ball	2016	USA	Builds on Ball et al.'s MKT framework [4] and Cohen et al.'s instructional triangle [38]
Zaslavsky and Leikin	2004	Israel	Builds on Jaworski's "teaching triad model" for teacher knowledge [39]

Table 2. *Cont.*

Author(s)	Year	Origin	Research Purpose
Components of a Framework for MTE Knowledge (n = 5)			
Chavout	2008	USA	Proposes curricular knowledge for MTEs: knowledge of programs and materials, lateral and vertical curriculum knowledge, indications and contraindications of curricula
Chick and Beswick	2013	Australia	Proposes PCK for MTEs: knowledge of examples, curricula, student thinking, common misconceptions
Chick and Beswick	2017	Australia	Proposes PCK for MTEs
Felton-Koestler	2020	USA	Proposes MTE knowledge for sociopolitical mathematics teaching: knowledge of sociopolitical issues and sociopolitical curriculum
Olanoff	2011	USA	Proposes components of MTE knowledge for teaching multiplication and division: multiple representations of topics, how topics relate, how to identify student learning goals in relation to topics, and how to design and use topic assessments

7.1. Comprehensive Frameworks for MTE Knowledge

In one of the earliest articles we reviewed, Mason conceptualized MTE knowledge not as different types of knowledge or knowledge domains, but rather as different levels of awareness that encompass the locus, focus, and form of teachers' attention in their teaching practice [40]. Through a self-reflective analysis, he proposed three levels of awareness, each of which encompasses the previous one: (a) awareness-in-action, which involves an awareness of one's actions and own learning, i.e., the development of one's knowledge of mathematical content; (b) awareness-in-discipline, which includes an awareness of one's awareness-in-action, i.e., the development of one's mathematical knowledge needed for teaching so that one can develop others' knowledge of mathematical content; and (c) awareness-in-counsel, which encompasses an awareness of how to develop awareness-in-discipline in others, i.e., the development of one's knowledge for teaching teachers in order to develop another's mathematical knowledge for teaching. Implicit within each level is a consciousness of one's awareness at that level. For Mason, MTE knowledge is synonymous with awareness -in -counsel, as MTEs enhance different levels of PTs' awareness (as learners and future teachers) [40]. Notably, unlike other frameworks, Mason conceptualizes MTE knowledge as an awareness of or attention to elements of teaching practice as opposed to conceptualizing MTE knowledge as the possession of certain information and skills about teaching practice.

Several other articles structure MTE knowledge in relation to teacher knowledge in similar ways. For example, both Zaslavsky and Leikin [41] and Perks and Prestage [42] include an existing teacher knowledge framework as one of the many subdomains of MTE knowledge in their frameworks and propose analogous knowledge subdomains in their MTE knowledge framework. For example, Zaslavsky and Leikin [41] expanded Jaworski's teacher knowledge framework [39], the "teaching triad of mathematics teachers", which highlights the interactions among three elements of teaching: challenging content for students, management of students' learning, and sensitivity to students. In their "teaching triad of MTEs" framework, Zaslavsky and Leikin created analogous terms for the knowledge required to teach teachers: challenging content for mathematics teachers (i.e., content to be learned), management of mathematics teachers' learning (i.e., how to support teachers during instruction), and sensitivity to mathematics teachers (i.e., understanding who teachers are as learners) [41]. Zaslavsky and Leikin considered Jaworski's "teaching triad for mathematics teachers" to be entirely contained within challenging content for mathematics teachers and thus a subdomain of MTE knowledge.

Similarly, Perks and Prestage proposed the "teacher-educator knowledge tetrahedron" [42], positioning the "teacher knowledge tetrahedron" framework [37] as a subdomain of MTE knowledge. Both frameworks include four subdomains: teacher knowledge, professional traditions, practical wisdom, and learner knowledge. However, in the teacher-educator knowledge tetrahedron, the learners in question are PTs and the teachers in question are MTEs. For Perks and Prestage, teacher educator knowledge refers to the knowledge MTEs develop over time as they teach PTs, professional traditions refer to knowledge of teacher preparation coursework and research on mathematics teacher education, practical wisdom refers to the tasks and activities used with PTs, and learner knowledge refers to the content that PTs need to understand [42]. Notably, the teacher knowledge tetrahedron [37] is entirely contained within the learner knowledge subdomain of their teacher-educator knowledge tetrahedron. Thus, like Zaslavsky and Leikin [41], Perks and Prestage conceptualize MTE knowledge as not only including the knowledge that teachers need to know, but also knowledge of other teaching elements [42].

Five additional frameworks draw specifically on Ball and colleagues' MKT teacher knowledge framework to conceptualize MTE knowledge [4]. All structure the teacher knowledge framework as entirely contained within an MTE knowledge subdomain and, in some instances, propose analogous MTE knowledge subdomains. For example, Olanoff et al. and Castro Superfine et al. proposed a conceptualization of MTE knowledge [43,44] that similarly extends from and is connected to the subdomains of MKT [4]. They posit that, similar to how MKT is composed of subject matter knowledge and PCK, MTE knowledge includes subject matter knowledge for MTEs (including MKT) and PCK for MTEs (for facilitating PTs' learning of MKT). They conceptualize MTE subject-matter knowledge as comprised of three subdomains analogous to those comprising subject matter for teachers: MTE knowledge of MKT (Ball et al.'s MKT framework [4]), MTE specialized content knowledge (mathematical content knowledge that is specific to developing PTs' MKT), and knowledge at the mathematical horizon for PTs. They conceptualize MTE PCK as being composed of three subdomains analogous to those comprising PCK for teachers: knowledge of content and PTs (i.e., MTE knowledge of content and students), knowledge of content and teaching PTs (i.e., MTE knowledge of content and teaching) and knowledge of curricula for PTs (i.e., MTE knowledge of curricula).

In their conceptualization of MTE knowledge, referred to as Mathematical Knowledge for Teaching Future Teachers (MKT-FT) [45], Hauk and colleagues also expanded upon the MKT framework [4] while integrating it with another teacher knowledge framework from their previous work [46]. Their previous framework included subdomains of university-based MTE PCK. They argued that akin to how teachers require specialized mathematical knowledge for teaching students, MTEs require specialized knowledge specific to supporting PTs in developing MKT. The MKT-FT framework includes four subdomains: knowledge of discourses (i.e., knowledge of communicating about MKT and the teaching of mathematics in different instructional situations), knowledge of content and teaching (i.e., knowledge of teaching mathematics to PTs and attention to MKT in making instructional decisions), knowledge of curricula (i.e., materials and resources used to teach PTs), and knowledge of content and students (i.e., knowledge of how PTs learn and understand content) [45]. Their previously developed model for teacher knowledge [46] is entirely contained within this last subdomain, which they describe as including knowledge of how to support PTs' development of MKT, as well as knowledge of how to engage PTs in learning to unpack mathematical ideas in ways needed for teaching students [45].

Chauvot also drew on the MKT framework [4], as well as Shulman's framework for teacher knowledge [26], and proposed analogous MTE knowledge subdomains from both frameworks [18]. Specifically, she proposed subject matter content knowledge, PCK, and curricular knowledge that parallel knowledge subdomains for teachers described by Shulman [26]. Chauvot's model expanded Shulman's work by placing all three of his subdomains of knowledge for teaching within the subject-matter content knowledge of MTEs. She also extended several of the MKT domains [4] to MTE knowledge, including knowledge of how to develop PTs' specialized knowledge, PCK, and knowledge of how to engage PTs with content in ways that are connected to teaching. Chauvot also included a knowledge of context subdomain, which is an understanding of contextual factors affecting the teachers with whom they work (e.g., standards and policies for teacher preparation).

Shaughnessy and colleagues similarly drew on the MKT framework [4], as well as another framework (albeit not a teacher knowledge framework per se). They expanded Cohen and colleagues' instructional triangle for teaching (i.e., interactions among teachers, students, and content) [38], positioning it as the content knowledge needed by MTEs [35]. Furthermore, like Chauvot [18], they proposed analogous subdomains of MTE knowledge that parallel those of the MKT framework [4], arguing that MKT is a subdomain of MTE knowledge. These subdomains include the following: MTE common content knowledge (e.g., knowledge of how to explain multi-digit subtraction that allows PTs to engage in this instructional practice), MTE knowledge of content and students (e.g., knowledge of common errors that PTs tend to make when engaging in instructional practices like regrouping using base-10 blocks), and MTE knowledge of content and teaching (e.g., knowledge of the types of tasks, representations, etc., that are useful in helping PTs learn elements of mathematics teaching) [35].

Escudero-Avila and colleagues drew on past research about mathematics teachers and teaching to propose seven subdomains of MTE knowledge: (1) mathematical knowledge (i.e., MKT), (2) knowledge about teachers' PCK (i.e., theories of teaching and learning standards), (3) knowledge about mathematics teaching practices and skills, (4) knowledge about professional identity (i.e., PT conceptions and beliefs), (5) knowledge of the features of the professional development of mathematics teachers, (6) knowledge of teaching the content of initial math teacher education programs (i.e., PT coursework and relevant activities), and (7) knowledge of the standards of math teacher education programs [47]. While not proposed as a framework per se, Escudero-Avila and colleagues' subdomains share many similarities with subdomains in other MTE knowledge frameworks included in our review. Notably, the authors did not propose a structure for how to relate the various knowledge subdomains.

7.2. Components of a Framework for MTE Knowledge

Five articles listed in Table 2 do not include comprehensive frameworks for MTE knowledge but either conceptualize components of an MTE knowledge subdomain or propose a new subdomain of MTE knowledge. For example, Chick and Beswick conceptualize components of MTE PCK, building their descriptions of PCK for MTEs from descriptions of the PCK teachers require [48,49]. They identified several components of MTE PCK, including knowledge of examples, curricula, student thinking, and common misconceptions. Using vignettes from the first author's practice, they provided evidence of the existence of the various components of MTE knowledge, illustrating ways in which such knowledge is leveraged as an MTE teaches PTs. Similarly, Chauvot conceptualized four components of curriculum knowledge for MTEs, including (a) knowledge of programs and materials (e.g., different models for teacher preparation, textbooks, and materials for use in courses for PTs), (b) knowledge of indications and contraindications of curricula (e.g., use of curricula or program materials in particular circumstances and the effectiveness of curriculum programs), (c) lateral curriculum knowledge (e.g., knowledge of other courses PTs are enrolled in), and (d) vertical curriculum knowledge (e.g., knowledge of coursework that precedes and follows current courses in which PTs are enrolled) [50]. Chauvot concluded by highlighting the use of MTE curricular knowledge in current studies of MTEs' professional learning, arguing for the centrality of curricular knowledge in MTEs' work. Olanoff also conceptualized components of a larger MTE knowledge domain, with a particular focus on the knowledge components leveraged while teaching a particular concept to PTs [51]. These components include knowledge of (a) multiple representations of the topics, how the representations relate to other topics, and which representations best support PTs in making connections; (b) how to set specific goals for student learning; and (c) how to design and use assessments effectively. Finally, Felton-Koestler proposed knowledge for sociopolitical mathematics teaching—a knowledge subdomain that he argues teachers at all levels need, including MTEs, for mathematizing sociopolitical issues in order to address issues of equity and social justice in their work [52]. Felton-Koestler describes this subdomain as including knowledge of sociopolitical issues and a broader sociopolitical curriculum for teachers.

8. Empirical Studies of MTE Knowledge

As a result of our review, we categorized 10 articles as empirical studies of MTE knowledge in Table 3. (We use the term "empirical" to refer to articles that follow a more traditional approach to research reporting, with a research question(s), methods, and results or findings.) Throughout the following sections, we discuss the ways in which authors approached the study of MTE knowledge and the findings of their work.

Table 3. Empirical studies of MTE knowledge (n = 10).

Author(s)	Year	Origin	Research Question(s)
How MTE Knowledge is Leveraged in Teaching PTs (n = 6)			
Castro Superfine and Li	2014a	USA	What forms of knowledge do MTEs use, and how is this knowledge different from that of teachers?
Jankvist, Clark, and Mosvold	2020	Denmark, USA, and Norway	What knowledge do MTEs leverage when describing how they would engage in certain tasks of teaching with PTs?
John	2002	UK	What are the knowledge and expertise that teacher educators hold and use?
Veselovsky	2017	USA	What goals for teaching and learning do MTEs develop in their teaching of content courses? What practices do they foster and why do they draw on particular forms of knowledge?
Zazkis and Mamolo	2016	Canada	How can the work of an MTE be explained in terms of levels of awareness [40] and knowledge of the mathematical horizon [53]?
Zopf	2010	USA	What is the work of teaching math to teachers, and what are the knowledge demands entailed by this work?
Self-Studies of MTE Knowledge in Practice (n = 4)			
Masingila, Olanoff, and Kimani	2018	USA	When MTEs reflect on their teaching in a community of practice, what types of MKTT do they use and develop?
Mendoza Álverez, Rhoads, and Jorgensen	2020	USA	How is MTEs' knowledge developed as they implement and revise tasks in math courses for PTs?
Muir, Wells, and Chick	2017	Australia	What aspects of MTE PCK are evident when MTEs observe each other's practice? In what ways do MTEs make their teaching practice explicit to PTs?
Muir, Livy, and Downton	2021	Australia	How does the knowledge needed by MTEs differ from that required by schoolteachers? To what extent is the Knowledge Quartet [54] applicable to MTEs?

8.1. Studies of How MTE Knowledge Is Leveraged in Teaching PTs

Zopf [33], Castro Superfine and Li [55], and Jankvist and colleagues. [56] analyzed the knowledge leveraged by MTEs as they teach PTs, drawing on different artifacts of teaching and interviews with MTEs to articulate ways in which MTE knowledge is distinct from the knowledge leveraged by teachers in their practice. Zopf [33] and Castro Superfine and Li [55] did so by analyzing instances in which MTEs teach tasks that develop PTs' specialized content knowledge. For example, using observations of and interviews with two different MTEs, Zopf identified three ways in which teaching PTs is different from teaching students: both the content and learners are different and, most notably, the purposes of instruction are different [33]. As such, the knowledge MTEs use to develop PTs' specialized content knowledge is unique. Castro Superfine and Li studied artifacts (i.e., videotaped classes, lesson plans, and slides) taken from several iterations (and multiple instructors) of a content course for PTs to illustrate instructional examples in which MTEs use their knowledge in practice [55]. In one example, during a discussion of students' errors related to multiplication, the authors described how MTEs not only need to have knowledge of common student errors but also knowledge of instructional strategies teachers might use to determine the sources of such errors. In doing so, the MTEs used this knowledge to both elicit PTs' understanding of these errors and connect PTs' learning to the work of teaching. Using course artifacts from and interviews with participating MTEs, Jankvist and colleagues analyzed the knowledge MTEs leveraged as they described how they would engage in certain tasks of teaching PTs (e.g., selecting examples and managing enactment of tasks) [56]. Collectively, by drawing on various artifacts of teaching and interviews with MTEs, these studies illustrate empirical approaches to studying how MTEs use their knowledge in practice when teaching PTs.

While not particularly focused on the uniqueness of MTE knowledge, Veselovsky analyzed the relationships among MTEs' knowledge in practice, their goals for PTs' learning, and their instructional practices [21]. Using classroom observations, interviews, and course artifacts, she developed case studies of four MTEs teaching content courses for PTs. Although Veselovsky found considerable alignment among individual MTEs' teaching and learning goals and their demonstrated knowledge, her cross-case analysis also identified notable differences. For example, while three MTEs indicated that one main teaching goal was to activate PTs as resources for each other, their purposes for doing so were different. One MTE saw all learners as having valuable knowledge, while another saw mathematical knowledge as a tool for social improvement, arguing that everyone should have access to mathematics. Similarly, John demonstrated the dynamic relationship between MTE knowledge and practice in his exploratory study of teacher educators in different disciplines [57]. Relevant to this review, John presented the case of Beth, an MTE, and characterized the ways in which she drew on her knowledge and experience in teaching PTs. Using interviews and classroom observations, John described how Beth leveraged her knowledge of research in mathematics education in her instructional interactions with PTs. In describing Beth's practice, John characterized the practice of teaching PTs as that of reorienting, that is, supporting PTs in reconceptualizing for themselves what it means to learn and teach mathematics. Together, Veselovsky [21] and John [57] demonstrated that an analysis of the knowledge MTEs use in their practice can reveal insights into the relationship between MTEs' knowledge and their instructional goals with PTs.

In contrast to focusing on MTE knowledge generally, Zazkis and Mamolo analyzed a specific subdomain of MTE knowledge and how it was used by an MTE in practice [58]. Drawing on Mason's levels of awareness [40], the authors proposed MTE knowledge of the mathematical horizon (see [4]), which they define as MTEs' use of mathematical knowledge beyond that which is required in school curricula. Using a particular instructional interaction, the authors focused on an MTE's response to a "disturbance," an instance in which an unexpected mathematical solution emerged during a class discussion to which the MTE did not have a suitable response. In their analysis, they described how the MTE reflected on the disturbance and revisited the mathematics underlying the unexpected solution, eventually connecting the solution to mathematical ideas that resided beyond the curriculum of the course. In doing so, they illustrated an MTE's knowledge of the mathematical horizon as an awareness of the periphery of a mathematical concept, understanding that the concept lies within a larger mathematical structure that is not necessarily accessed during teaching.

8.2. Self-Studies of MTE Knowledge in Practice

The remaining four empirical studies in our review analyzed how the authors themselves, as MTEs working within a larger collaborative community of colleagues, use their knowledge in practice teaching PTs. Muir and colleagues, for example, focused on a particular subdomain of MTE knowledge and explored the PCK that MTEs use when teaching PTs [59]. Two of the authors observed each other's courses for PTs across two semesters and engaged in pre-planning and post-observation discussions about their practice and collected artifacts of PT learning. The authors used Chick and Beswick's framework for MTE PCK [48] to identify instances in their data sources where the MTEs drew on MTE PCK. Once identified, the authors applied the Knowledge Quartet framework from Rowland and colleagues [54] to analyze those instances. While acknowledging similarities between the knowledge teachers require and the knowledge the MTEs used in their study, the authors illustrated the unique nature of MTE PCK, demonstrating ways in which MTEs support PTs' development of PCK.

In their study of MTEs within a community of practice, Masingila and colleagues examined ways in which MTEs develop their knowledge for teaching teachers by reflecting on the process of teaching content courses for PTs [60]. Drawing on Zaslavsky and Leikin's notion that MTEs develop knowledge through teaching [41], the authors focused on the types of MTE knowledge that they used and gained while supporting PTs' development of MKT. Over the course of two semesters, the authors each taught sections of mathematics content courses for PTs focused on problem solving. Using transcripts from planning and debriefing meetings, observations of the courses, and reflective memos of their own practice, the authors identified three tasks of teaching PTs wherein the MTEs used and further developed their own knowledge. For example, with the task of identifying goals for PTs, the MTEs considered the concepts most important for PTs to engage with to develop MKT and how to facilitate the learning of these concepts. The authors argued that participating in a community of practice focused on teaching courses for PTs provided opportunities for them to reflect on their own practice and make explicit how they were drawing on their knowledge as MTEs in their practice.

Mendoza Álverez and colleagues studied how their own knowledge as MTEs was further developed as they implemented and revised various tasks in their mathematics courses for PTs [61]. Building upon Chick and Beswick's conceptualization of MTE PCK [49], the authors discussed the central role that MTE PCK plays in the work of teaching PTs. Drawing on classroom observations, PT artifacts, and their own reflections, the authors illustrated the evolution of tasks implemented across different courses and described their efforts to improve their facilitation of those tasks over time. For example, they discussed how their reflections on how PTs interacted with tasks led to them changing their facilitation moves and the types of questions they asked to ensure that PTs maintained a high level of cognitive challenge. In doing so, they highlighted how their own MTE PCK [49] was enhanced as they implemented and revised their tasks for PTs.

In their study, Muir and colleagues focused on the applicability of the Knowledge Quartet [54] framework to understanding MTE knowledge in practice [59]. Like other frameworks for teacher knowledge, the Knowledge Quartet includes domains that address knowledge of content, pedagogy, and instructional routines (e.g., lesson planning). Using a case study methodology, the authors examined the knowledge an MTE used as they co-taught, with a schoolteacher, a PT course focused on mathematics pedagogy. Drawing on lesson plans, class observations and individual interviews, the results of their analysis indicated similarities in the type of knowledge used by the MTE and schoolteacher and highlighted important differences. In particular, Muir and colleagues pointed to an example in which the MTE demonstrated knowledge of the theoretical rationale underlying certain practices (e.g., why certain models are more efficacious for teaching place value) [59]. Their analysis also indicated that the MTE drew on knowledge of how to model various instructional practices for PTs and to articulate why and how such practices were supportive of student learning. Overall, the study by Muir and colleagues points to the uniqueness of the knowledge MTEs use in teaching PTs. Moreover, while not extending the subdomains of the Knowledge Quartet framework to MTEs, the authors demonstrate the applicability of a teacher knowledge framework to analyzing MTE knowledge in practice.

Taken together, the articles on MTE knowledge provide empirical evidence of the uniqueness of MTE knowledge and illustrate ways in which this knowledge is distinct from teacher knowledge based on analysis of artifacts of teaching PTs. Moreover, the articles in this section highlight the unique forms of knowledge MTEs leverage when teaching PTs through the interconnections among their knowledge, goals, and instructional decision making. Notably, all of these articles approached the study of MTE knowledge through qualitative analyses of teaching and learning artifacts, observations, and interviews with and reflections from MTEs.

9. Contexts for MTE Knowledge Development

We placed nine articles in the Contexts for MTE Knowledge Development category, as shown in Table 4. Two articles describe the impact of MTEs' research activities on their development of knowledge, six share the design of specific professional learning opportunities for MTEs, and one discusses the ways in which conducting self-studies can lead to MTE knowledge development.

Table 4. MTE knowledge development (n = 9).

Author(s)	Year	Origin	Research Question(s)
Conducting Research to Develop MTE Knowledge (n = 2)			
Peled and Hershkovitz	2004	Israel	In what ways do various phases of research conducted by MTEs afford them learning opportunities, and what conditions contribute to their learning?
Rowland, Turner, and Thwaites	2014	UK	In what ways do MTEs experience changes in their awareness and practice as a consequence of conducting research in math education?
Professional Learning Experiences to Develop MTE Knowledge (n = 6)			
Castro Superfine and Li	2014b	USA	What are MTEs' design decisions and goals for their math content courses? How do MTEs achieve these goals? What challenges do MTEs encounter as they carry out these goals?
Golding and Batiibwe	2021	UK and Uganda	What are the affordances and constraints in the design of a Mathematical Thinking and Instructional Technology course for MTEs? What impact did the course have on MTE knowledge?
Roesken-Winter, Schüler, Stahnke and Blömeke	2015	Germany and Norway	What was the impact of a year-long course for MTEs on MTE content knowledge and pedagogical content knowledge? To what extent did the design principles applied in the course become relevant for MTEs' work?
Shaughnessy, Garcia, Selling, and Ball	2016	USA	What knowledge/skills are demanded of MTEs to engage in practice-based teacher education? What structures are useful for supporting novice MTEs' learning of such knowledge/skills?
Szatjn, Ball, and McMahon	2006	USA	How can researchers design professional learning models for MTEs focused on developing their understanding of MKT for PTs?
Van, Mao, and Cnudde	2018	Cambodia	What is the impact of a professional development intervention on MTEs' PCK and content knowledge of rational numbers?
Developing MTE Knowledge Through Self-Study (n = 1)			
Taylan and da Ponte	2016	Turkey and Portugal	How does teaching and researching students support MTEs in developing MKTT?

9.1. Conducting Research to Develop MTE Knowledge

Two articles in our review described ways in which MTEs' research activities contributed to their own development of knowledge in practice. Rowland and colleagues provided a reflective account of their development as MTEs, describing changes in their awareness [40] and instructional practices as a result of conducting research on teacher knowledge [62]. Using narrative inquiry with elements of autobiography, the authors described instances where each drew on one or more dimensions of the Knowledge Quartet [54] in their work teaching PTs. For example, they described how PTs often selected mathematical representations based on superficial features without attention to the underlying mathematical ideas. To address this in their practice, the authors discussed how they drew from their research on teacher knowledge to inform their teaching of mathematical representations. The authors concluded that even though the focus of their research was on the knowledge development of PTs, their work resulted in their own professional development, as their enhanced understanding of teacher knowledge informed the content and design of their courses for PTs.

Similarly, Peled and Hershkovitz described a three-phase research study in which they worked with PTs on problem-solving tasks, reflected on what PTs learned, and then reflected on their own knowledge development as MTEs [63]. Drawing on both Zaslavsky and Leikin's model [41], which emphasizes the role of reflection in MTE professional learning, and Jaworski's model for MTE–learner relations [64], the authors reflected on three examples from their research and described the nature of their learning in each. In particular, they discussed their dual attention to observing PTs' engagement with the tasks (as MTE instructors) and their efforts to design tasks in ways that enable data collection to support their research goals (as MTE researchers) [63]. For example, they described revising their plan for implementing a particular task, basing their decisions on two elements of Jaworski's model (i.e., sensitivity to learners and mathematical challenge) [64]. Peled and Hershkovitz provide an example of MTEs learning from and in teaching, as they ultimately enhanced their own theoretical understanding of mathematical tasks by designing tasks and reflecting on PT learning [63].

9.2. Professional Learning Experiences to Develop MTE Knowledge

Six of the articles we reviewed described the design of professional learning experiences for MTEs and, in some cases, presented descriptive analyses of ways in which the designs supported MTE professional learning. In their work, Shaughnessy and colleagues focused on ways of supporting novice MTEs in learning to teach PTs [35], arguing that MTEs require specialized knowledge for teaching MKT [4] to PTs. The authors describe various support structures within their weekly planning group that support novice MTE learning. One structure includes rehearsals, wherein an experienced MTE roleplays a PT who is explaining core content, a novice MTE runs the rehearsal, and another experienced MTE provides feedback to the novice MTE on the rehearsal. This structure provides opportunities for novice MTEs to develop and receive feedback on their practice and to further develop their specialized knowledge for teaching PTs.

While not specifically focused on novice MTEs, Szatjn and colleagues [65] and Castro Superfine and Li [66] described designs and illustrated implementations of more formal professional learning experiences for MTEs. These experiences included either monthly workshops or intensive week-long summer retreats during which MTEs analyzed live or recorded observations of MTEs teaching PTs. With the overall aim of enhancing their own knowledge in practice, MTEs engaged in professional inquiry of artifacts from MTEs teaching content courses. In both designs, the authors focused on MTEs' development of the specialized content knowledge needed by MTEs for teaching PTs.

Similarly, Van and colleagues described a professional learning intervention focused on improving MTEs' content knowledge and PCK of rational numbers [67]. The intervention consisted of three multi-day modules focused on understanding rational numbers and related representations and assessments. Following the modules, MTEs taught rational number-focused lessons and received mentoring around those lessons. Through a pre-/post-test design, together with lesson observations of MTEs, the authors found that a majority of participating MTEs significantly increased their PCK, including their ability to assess PTs and address PT misconceptions of rational numbers. The authors argue for the need for more professional learning experiences for MTEs to further develop their own knowledge, as few exist.

The remaining two articles described the design and impact of courses for MTEs. Golding and Batiibwe discussed a Mathematical Thinking and IT (Instructional Technology) course designed to enhance MTEs' technological, mathematical, and mathematics pedagogical knowledge through various types of activities [68]. Using data from tests, course assignments, and individual interviews, their results indicated that the participating MTEs demonstrated moderate improvements in their mathematical knowledge, and the MTEs who showed improvement reported feeling confident in their ability to apply ideas from the course to their local contexts. Roesken-Winter and colleagues similarly described the impact of a year-long course for MTEs on MTE content knowledge and PCK [69]. The multiple course modules ranged in focus from mathematics content and PCK to behavior management and technology skills. Data from questionnaires and individual interviews indicated significant development of MTE knowledge, among other positive outcomes. The authors also measured the potential sustainability of the course on MTE outcomes (i.e., 6 months post-course) and found similar positive gains in both types of MTE knowledge. Together, these articles demonstrate the potential of focused coursework for enhancing MTE knowledge in practice.

9.3. Developing MTE Knowledge through Self-Study

While several researchers have employed self-study to analyze MTE practice, only one article we reviewed explicitly focused on how self-study can enhance MTEs' development of knowledge. Taylan and da Ponte focused on the first author's professional learning from her practice teaching a group of elementary students, as part of a larger university-school research program [17]. In particular, the authors focused on ways in which the MTE drew on her PCK in her instructional interactions with the students. Using videotaped classroom observations and the MTE's reflections on particular instructional interactions, Taylan and da Ponte illustrated ways in which the MTE refined certain aspects of her own PCK over time. In doing so, the authors argued that analyzing the practices MTEs use while teaching young students can provide opportunities for MTEs to make important connections between research and practice, which they can in turn make explicit when teaching PTs.

Overall, the research on MTE professional learning highlights the prevalence of self-study and the use of reflective analysis to develop MTEs' knowledge in practice. Less commonly reported was the conduction of research in mathematics teacher education as a form of professional learning for MTEs. Most notable were the six articles that described formal professional learning opportunities for MTEs [35,65–69]. What this review makes clear is that the majority of opportunities for professional learning are largely those that are initiated by MTEs themselves in the form of self-studies or collaborations with colleagues.

10. Discussion

There is a general consensus within the teacher education community that the knowledge MTEs require in their work with PTs includes not only the knowledge that teachers need but also unique elaborations of that knowledge. In fact, many of the articles we reviewed conceptualize MTE knowledge as an extension of teacher knowledge. That is, these frameworks not only position teacher knowledge as a subdomain of MTE knowledge but also partition MTE knowledge into subdomains similar to those found in frameworks for teacher knowledge. In our previous work, we proposed a fractalization metaphor to describe the ways in which various articles conceptualize MTE knowledge, as the visualizations of many of these frameworks resemble part of a fractal [43,44,70]. We use the term *fractalization* to refer to the process by which one knowledge component or subdomain is entirely contained within a larger knowledge component or subdomain, where the larger component or subdomain is analogous in structure to the smaller one.

11. Conceptual and Structural Similarities across MTE Knowledge Frameworks

Many of the MTE knowledge frameworks we reviewed build on existing frameworks for teacher knowledge and, in many instances, represent fractalizations of teacher knowledge frameworks (e.g., the teaching triad [38], the instructional triangle [39], and the teacher knowledge tetrahedron [37]). In other words, many frameworks for MTE knowledge include a teacher knowledge framework in its entirety as one of their subdomains. Furthermore, the researchers conceptualized certain subdomains (and related components) of MTE knowledge as being "meta" forms of analogous teacher knowledge subdomains (e.g., [18,41,42,45]). For example, Shaughnessy and colleagues [35], Olanoff and colleagues [44], and Castro Superfine and colleagues [43] (proposed MTE knowledge subdomains analogous to the MKT framework from Ball and colleagues [4], including MTE common content knowledge (which contains MKT), MTE knowledge of content and students, and MTE knowledge of content and teaching. In other words, as teachers of teachers, MTEs require similar types of knowledge needed by teachers, but MTEs need knowledge of these subdomains in ways that are specific to teaching PTs. We found such structural similarities across many of the frameworks we reviewed. Such a structure reinforces a conceptualization of MTE knowledge not only as an extension of teacher knowledge but also as including knowledge subdomains that are uniquely different from teacher knowledge.

Despite overall structural similarities, there are important conceptual differences in the subdomains of the frameworks we reviewed. Broadly speaking, all of the knowledge frameworks in our review include some or all of the four main subdomains representing extensions of teacher knowledge domains (e.g., [4,26]): knowledge of content, knowledge of curriculum and context, knowledge of PTs, and knowledge of ways of supporting PT learning. However, some frameworks include subdomains unique to MTEs, such as Zaslavsky and Leikin's sensitivity to mathematics teachers (PTs) [41]. Considering the uniqueness of PTs as a population of learners, this knowledge includes understanding that PTs often enter teacher preparation programs with limited conceptual understandings of math. As such, the work for MTEs is to support PTs' relearning of math, which involves PTs ultimately reconstructing their previously developed knowledge of mathematics [43,71]. Further, Chauvot included MTEs' knowledge of mathematics education research as part of the knowledge needed to effectively prepare PTs [18]. In fact, Chauvot posited that knowledge of research in mathematics education underlies all the other knowledge subdomains MTEs require in their work with PTs (e.g., knowing the research on how children learn different concepts can inform MTEs' design of content courses for PTs). This echoes other researchers who describe conducting research in mathematics teacher education as a form of professional learning (e.g., [62]). Notably, these different subdomains are unique to MTEs and arguably do not have analogous subdomains in a teacher knowledge framework. In addition, there were important differences in the grain size at which the researchers whose work was considered in this review conceptualized MTE knowledge. While the majority only identified subdomain levels using the four main subdomains described above, a few deconstructed their subdomains into components. For example, Chauvot described different components of MTE curricular knowledge (e.g., lateral and vertical curricular knowledge) [20], whereas Castro Superfine and colleagues specified components of MTEs' content-specific knowledge (e.g., specialized content knowledge and knowledge of content and teaching) [43]. More work needs to be done to theorize components of the four main subdomains of MTE knowledge and empirically explore the ways MTEs leverage such knowledge components in their teaching.

Another theme that emerged is the process by which MTE knowledge has been studied and conceptualized, which has largely been from a knowledge-in-practice perspective. Through an analysis of various artifacts of practice and reflections on MTEs' work with PTs, researchers applying a knowledge-in-practice perspective to the work of MTEs highlight the types of knowledge leveraged as they teach PTs, reinforcing the dynamic relationship between knowledge and practice. A majority of the articles on MTE knowledge described a self-study process wherein one or more of the authors reflected on and described the types of knowledge they leveraged in their work with PTs (e.g., [58–60]. Such a process is productive for understanding the types of resources (e.g., experiences, beliefs, and knowledge) that impact MTE practice to support drawing on different types of knowledge considering their expertise. However, more work needs to be done to explicate the analytic processes employed in research on MTE knowledge so that others can employ similar methods and contribute to the knowledge base. Moreover, while self-studies (i.e., research by MTEs) provide unique insights into the nature of MTE knowledge, the research base would be strengthened by research on MTEs to corroborate and further specify the various MTE knowledge subdomains and components. While not included in our review, Goos' analytic framework provides a useful starting point for studying MTE knowledge in practice by considering the interactions among MTE knowledge, beliefs, and contexts and the ways in which these interactions manifest in the various knowledge subdomains leveraged by MTEs [72].

Another theme that emerged was the application of self-study approaches by researchers to empirically study MTE knowledge and its subdomains. For example, Muir and colleagues, who examined MTE PCK [59]; Zazkis and Mamolo, who focused on MTEs' knowledge of the mathematical horizon [58]; and Masingila and colleagues [60] and Bergsten and Grëvholm [32], who focused on MKTT more broadly, all employed a version of self-study, where one or more of the authors was the object of study while the other author(s) served as researcher(s). The methods employed for the "object" of study ranged from general qualitative inquiries into and analyses of MTEs' reflections and analyses of artifacts of teaching practice (e.g., observations, meeting notes, PT work samples, and lesson plans) to retrospective analyses of their own practice with PTs. Similar to those in the research base for frameworks for MTE knowledge, self-studies of specific aspects of MTE knowledge provide unique insights that other approaches cannot offer. However, Adler and colleagues argue that, in the context of such self-studies, researchers need to take a critical stance with respect to their practice and include external observers to strengthen the arguments being made [73].

12. Expanding the Contexts for MTE Knowledge Development

Despite the centrality of MTEs in teacher education, little is known about the nature of professional learning opportunities for them to develop and enhance the knowledge needed to teach PTs. This is particularly important as MTEs who teach PTs include mathematicians, graduate students, mathematics educators, and classroom teachers who have varying professional backgrounds and are often not professionally prepared for the work of teaching teachers [24]. There were few articles in our review that described formal opportunities for MTE professional learning that aimed to develop MTE knowledge (e.g., [65,66]). By formal we mean professional learning opportunities that mirror those for teachers, such as workshops, coursework, or summer institutes. Within these articles, professional learning took the form of either coursework (e.g., [68,69]) or structured workshops and experiences focused on MTE knowledge development (e.g., [35,65,66]). In these examples, the authors made explicit the designs of the coursework and professional learning opportunities, illustrating how MTE knowledge can be further developed through inquiry into the practice of teaching PTs. Indeed, there is a great need for the development of additional professional learning opportunities for MTEs such as these [5,24]. Goodwin and colleagues found that the majority of the teacher educators they surveyed learned to teach PTs "on the job", with no formal preparation for this work, as there are currently few, if any, graduate programs in the US devoted exclusively to the professional preparation of teacher educators [74]. One option for supporting the professional learning of MTEs might be to enlist experienced MTEs to develop graduate-level programs for preparing novice MTEs to teach PTs. In our review, the work of Shaughnessy and colleagues was most closely related to graduate-level programs—in this case, for novice MTEs [35]. In addition, though not a formal professional learning experience, Masingila and colleagues described their efforts to understand how MTE knowledge is developed when teaching MKT to PTs [60]. Their study involved two novice MTEs observing one more experienced MTE.

Across the articles we reviewed, the use of self-study was a common form of professional learning for MTEs. While our review only included one example of how self-study can be leveraged to enhance MTE knowledge (e.g., [17]), several articles that were not included in our review, because they did not focus explicitly on MTE knowledge, employed self-study methodologies to enhance MTEs' instructional practice (e.g., [16,75]). These articles, taken together with other recent scholarship on MTE learning (e.g., [11,76]), highlight how the use of self-study to support MTE professional learning has grown considerably in recent years. In all of these articles, the researchers (who were also participants) took a knowledge-of-practice perspective to analyze their own practice. Again, such "inside" perspectives are useful for understanding the development of MTE knowledge but should be complemented by "outside" perspectives, where researchers include external observers to promote a more critical stance [73].

13. Conclusions

While our review indicates a growing research base that builds on research related to teacher knowledge, it is fragmented in many ways. Nevertheless, if we are to pursue the process of building a knowledge base for MTEs, we must assume, like Lampert and Graziani, that there is some knowledge about teaching that is relevant across contexts and that it can be "taught" and "learned" [77]. By "taught", we do not mean to imply a purely didactic or transmission model of teaching and learning but rather suggest that a community of MTEs can supply useful information, establish an environment, and arrange experiences conducive to providing MTEs with opportunities to further develop their knowledge. In fact, research like that conducted by Zaslavsky and Leikin [41] offers evidence that knowledge in practice can be developed and shared within a community of MTEs in such a way that supports MTEs in learning, growing, and becoming more confident in their abilities to prepare effective teachers of mathematics.

We posit that there may be processes by means of which the field can identify knowledge that is generally useful across varied situations so that we might begin building a mutually agreed-upon knowledge base for MTEs. Such a knowledge base could inform the design and implementation of opportunities to improve the preparation and professional development of MTEs—a logical next step regarding the horizon of this landscape. However, it is worthy to note that Chapman has pointed out several potential issues surrounding the use of a category-based approach for conceptualizing MTE knowledge [78]. She states that "We are at the early stage of researching MTEs, and the category-based perspective makes sense at this point as a way to begin to understand MTE knowledge. However, it is important for research to explore other ways of understanding and representing MTE knowledge" (p. 415).

Our hope is that this review can contribute to the ongoing discussion related to MTEs and the development of their knowledge as this research base continues to grow and evolve.

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