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MULTICOMPLEX TAYLOR SERIES EXPANSION FOR COMPUTING HIGH-ORDER DERIVATIVES

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Abstract: Multicomplex Taylor series expansion (MCTSE) is a numerical method for calculating higher-order partial derivatives of a multivariable real-valued and complex-valued analytic function based on Taylor series expansion without subtraction cancelation errors. The implementation has been facilitated using Cauchy-Riemann matrix representation of multicomplex variables. In this paper, we show steps for finding these matrices and, in addition, that the number of appearances of the k^{th} derivatives follows the Pascal's triangle. Also, the situations where the MCTSE is not applicable is determined. Finally, we investigate the application of the method for complex-valued functions.

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1. Introduction

W.R. Hamilton in 1833, presented a paper to the Irish Academy in which he developed a formal algebra of real numbers, which is precisely the algebra of the complex numbers as usually understood today. In [3], he discovered quaternions, and the most well-known extension of the complex numbers is given by the quaternions, which are often used to represent rotations in three-

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dimensional space. However, quaternions are not commutative in multiplication, and this property prevents them from being suitable for computing partial derivatives. C. Segre [9], generalized the space of complex numbers, C_1, and defined an infinite set of algebras called bicomplex numbers, C_2, tricomplex numbers, $C_3, \ldots$, n-complex numbers, C_n. A complex number has the form $x_1+i_1x_2$ such that $x_1, x_2 \in C_0$ and $i_1^2 = -1$. A bicomplex number is a number of the form $x_1+i_1x_2+i_2x_3+i_1i_2x_4$ such that $x_1, \ldots, x_4 \in C_0$ and $i_1^2 = i_2^2 = -1, i_1i_2 = i_2i_1$. Since we know that $z_1 = x_1 + i_1x_2$ and $z_2 = x_3 + i_1x_4$ are complex numbers, another representation of bicomplex numbers is $z_1 + i_2z_2;\;\; z_1, z_2 \in C_1, i_2^2 = -1$. This representation is the reason why the elements $x_1+i_1x_2+i_2x_3+i_1i_2x_4$ are called bicomplex numbers. Depending on the context, these two representations may have significant effect on the development of the theory. Moreover, interesting problems arise from a comparison of results obtained from the two representation of bicomplex numbers. It is clear that C_0 is a subspace of C_1 and C_1 is a subspace of C_2. Mapping the bicomplex number $x_1+i_1x_2+i_2x_3+i_1i_2x_4$ into the point (x_1, x_2, x_3, x_4) in C_0^4, the space C_2 of bicomplex numbers is embedded in ${C_0}^4$. The bicomplex numbers $x_1+i_10+i_20+i_1i_20$ are isomorphic to the real numbers C_0 and for simplicity they are called real numbers; $0+i_10+i_20+i_1i_20$ and $1+i_10+i_20+i_1i_20$ are called zero and one and denoted by 0 and 1. The set $\{x_1+i_1x_2+i_20+i_1i_20 \mid x_1, x_2 \in C_0\}$ and C_1 are isomorphic under corresponding operations. Let ζ_1 and ζ_2 be elements in C_2. If $\zeta_1 \neq 0$, $\zeta_2 \neq 0$ and $\zeta_1\zeta_2 = 0$, then ζ_1 and ζ_2 are called divisors of zero. C. Segre [9], showed that there is an important difference between C_1 and C_2, the complex numbers form a field, but the bicomplex numbers do not since they contain divisors of zero, such as $e_1 = \frac{1+i_1i_2}{2}$ and $e_2 = \frac{1-i_1i_2}{2}$. We know that $e_1 \neq 0$ and $e_2 \neq 0$ but $e_1e_2 = 0$. Let ζ_1 and ζ_2 be elements in C_2. If ${\zeta_1}^2 = \zeta_1$ then ζ_1 is called an idempotent element. There are four idempotent elements in C_2, two of these idempotent elements, namely $(1 + i_1i_2)/2$ and $(1 - i_1i_2)/2$ play an important role since every element in C_2 has a unique representation as a linear combination of them. Multicomplex numbers were studied in detail in [2] and [8].

Computing first derivatives numerically has been extensively studied by mathematicians and engineers. However, computing higher order derivatives with high precision has not been studied in such depth, although it can have vast applications in science and engineering. Lantoine, et al. [5, 6] presents multicomplex Taylor series expansion, called MCTSE. They considered a small perturbation into the appropriate multicomplex number to gain precise results, and analyzed to verify and demonstrate the accuracy and efficiency of the algorithm. Recently, Vittaldev, et al. [10], computed the derivatives required for the second order Kalman Filter (SOKF) using the MCTSE method to account for nonlinearities in an estimation problem. MCTSE is proven to have many advantages, and it is therefore expected to be useful for any algorithm exploiting high-order derivatives, such as many non-linear programming solvers. The main advantages of the MCTSE method is that it is valid for any order of derivatives and partial derivatives with no subtractive cancelation error, and therefore the truncation error can be made arbitrarily small. In this paper, the MCTSE method has been used for the calculation of higher-order partial derivatives of a multivariable real-valued and complex-valued analytic function based on Taylor series expansion without subtraction cancelation errors. Steps for finding Cauchy-Riemann matrix representation of multicomplex variables has been shown here. In addition, we show that the number of appearances of the k^{th} derivatives follows the Pascal's triangle. Also, we find that the MCTSE is not applicable for nonanalytic functions. Finally, we investigate the application of the method for complex-valued functions.

A traditional method that is used for computing first derivatives is the finite difference method. When the source codes are black box, finite difference approximations are typically used to evaluate the derivatives. The step-size dilemma in the finite difference method forces a choice between a step size small enough to minimize the truncation error and a step size large enough to avoid significant subtractive cancelation errors. The complex step approximation is more accurate in engineering applications compared to finite difference because of no step size dependency, and not suffering from subtractive cancelation errors for the first derivatives. J. R. R. A. Martins and P. Sturdza [7], discussed that the complex step approximation is similar to the automatic differentiation (AD), but it is easier to implement. In fact the complex step approximation can be considered as an approximate implementation of automatic differentiation using the existing complex data structure in some computer languages. Two major sources of error affecting derivative approximations include: truncation error and subtractive cancelation error. Truncation error is basically related to elimination of higher order terms in the Taylor series expansion as it is used in derivative approximation. Smaller choice of the step size h can reduce the truncation error. Subtractive cancelation error is because of representation of numbers as a finite number of bits, and the significant digit in computer mathematical operations. This causes the difference of the numbers which are very close to each other to be practically zero, resulting in incorrect approximation to the value of the derivatives, as in finite difference and complex step methods. So we require small step sizes to minimize truncation error but that leads to subtractive cancelation errors. There are several difficulties that arise when trying to find the optimal step size. Typically, this optimal step size is problem dependent. It may also vary depending on the value of the independent variable for which the derivative is being computed. More importantly, for functions of multiple variables, it may be different for each variable. With finite-difference approximations, we cannot avoid truncation and subtractive cancelation errors simultaneously. Meanwhile, the MCTSE method contains no subtractive cancelation error, and therefore the truncation error can be made arbitrarily small. More information about finite difference and complex step approximations can be found in [1] and [7].

The MCTSE method shares with automatic differentiation the capability of computing systematically accurate partial derivatives with respect to desired input variables. However, a major difference is that the user has control of which components they want to compute by applying a perturbation step on specific imaginary directions and computing a series of multicomplex function evaluations. Therefore, the MCTSE method could be classified as semiautomatic differentiation.

W. Yu and M. Blair [12], introduced the dual number automatic differentiation (DNAD) method for computing the derivatives. They claimed that the advantages of this method are programming simplicity, extensibility, and computational efficiency. Although this method is simple to code for computing the first derivatives, the number of calculations in DAND method may not be less than the MCTSE method for computing high-order and partial derivatives. Also, DNAD cannot compute the high-order and partial derivatives in one step.

In Section 2, important theorems and corollaries that will be used in this paper and the procedure for finding the Cauchy-Riemann matrix representation of multi-complex variables to facilitate computing the high-order derivatives are introduced. In Section 3, we investigate the MCTSE method and show that the number of the k^{th} derivatives is the coefficient of binomial series. Also, we show that MCTSE is not applicable for nonanalytic functions. In Section 4, we investigate the use of multicomplex mathematics for the calculation of high-order derivatives of complex-valued functions. Finally, in Section 5, we concludes the important results of the paper.

2. Main Theorems and Corollaries

In this section we present some of the basis definitions and theories that will be needed and provide some examples to illustrate the concepts involved.

Definition 2.1. The set C_2 is defined by the following statements:

(1) $C_2 = \{x_1+i_1x_2+i_2x_3+i_1i_2x_4 \mid x_1,\ldots,x_4 \in C_0, i_1^2 = i_2^2 = -1, i_1i_2 = i_2i_1\}$.

(2) $x_1 + i_1x_2 + i_2x_3 + i_1i_2x_4 = y_1 + i_1y_2 + i_2y_3 + i_1i_2y_4$ if and only if $x_i = y_i, i = 1,\ldots 4$.

(3) Addition is the operation on C_2 defined by the following function,

$$\oplus : C_2 \times C_2 \to C_2,$$

$$(x_1 + i_1x_2 + i_2x_3 + i_1i_2x_4, y_1 + i_1y_2 + i_2y_3 + i_1i_2y_4) \mapsto$$

$$(x_1 + y_1) + i_1(x_2 + y_2) + i_2(x_3 + y_3) + i_1i_2(x_4 + y_4).$$

(4) Scalar multiplication is the operation on C_2 defined by the function,

$$\odot : C_0 \times C_2 \to C_2,$$

$$(a, x_1 + i_1x_2 + i_2x_3 + i_1i_2x_4) \mapsto$$

$$ax_1 + i_1ax_2 + i_2ax_3 + i_1i_2ax_4.$$

Observe that equality, addition, and scalar multiplication in C_2 are defined in terms of equality, addition, and multiplication of real numbers. This fact provides the basis for the proof of the following theorem.

Theorem 2.2. *The system $(C_2, \oplus, \odot)$ is a linear space over C_0.*

Proof. Theorem 2.2, [8]. □

Definition 2.3. The product $(x_1+i_1x_2+i_2x_3+i_1i_2x_4)\otimes(y_1+i_1y_2+i_2y_3+i_1i_2y_4)$ is the element in C_2 obtained by multiplying $x_1 + i_1x_2 + i_2x_3 + i_1i_2x_4$ and $y_1 + i_1y_2 + i_2y_3 + i_1i_2y_4$ as if they were polynomials and then using the relations $i_1^2 = -1$, $i_2^2 = -1$ and $i_1i_2 = i_2i_1$ to simplify the result. The following display exhibits this product and the final result:

$$(x_1 + i_1x_2 + i_2x_3 + i_1i_2x_4) \otimes (y_1 + i_1y_2 + i_2y_3 + i_1i_2y_4)$$

$$= (x_1y_1 - x_2y_2 - x_3y_3 + x_4y_4)$$
$$+ i_1(x_1y_2 + x_2y_1 - x_3y_4 - x_4y_3)$$
$$+ i_2(x_1y_3 - x_2y_4 + x_xy_1 - x_4y_2)$$
$$+ i_1i_2(x_1y_4 + x_2y_3 + x_3y_2 + x_4y_1).$$

Example 2.4. Definition 2.3, shows that:

$$(2 + i_14 - i_23 + i_1i_25) \otimes (6 - i_18 + i_23 - i_1i_27) = (18 - i_128 + i_256 + i_1i_252).$$

if z_1, z_2, w_1 and w_2 are elements in C_1, then

$$(z_1 + i_2z_2) \otimes (w_1 + i_2w_2) = (z_1w_1 - z_2w_2) + i_2(z_1w_2 + z_2w_1).$$

The above formula emphasizes once more the formal similarities of complex and bicomplex numbers.

Theorem 2.5. *The following statements describe properties of multiplication in C_2:*

(1) C_2 is closed under multiplication.

(2) Multiplication is associative.

(3) Multiplication is distributive with respect to addition.

(4) Multiplication is commutative.

(5) There is a unit element for multiplication; it is $(1 + i_10 + i_20 + i_li_20)$, which is usually denoted by 1.

Proof. Theorem 4.3, [8]. □

In Table 1, $i = i_1$, $j = j_1$, $k = i_1j_1 = j_1i_1$.

Definition 2.6. The spaces $C_n = 0, 1, \ldots, n, \ldots$ are linear spaces whose sets of elements, norms, and operations are defined as follows:

(1) $C_0 = \{x \mid x \in C_0\}$.
Addition and multiplication are the usual operations in C_0.

x	1	i	j	k
1	1	i	j	k
i	i	-1	k	-j
j	j	k	-1	-i
k	k	-j	-i	1

Table 1: Bicomplex multiplication.

(2) $C_1 = \{z \mid z = x_1 + i_1 x_2, x_1, x_2 \in C_0\}$.
Addition: $z_1 + z_2 = (x_1 + i_1 x_2) + (y_1 + i_1 y_2) = (x_1 + y_1) + i_1(x_2 + y_2)$.
Multiplication: $z_1 z_2 = (x_1 + i_1 x_2)(y_1 + i_1 y_2) = (x_1 y_1 - x_2 y_2) + i_1(x_1 y_2 + x_2 y_1)$, $i_1^2 = -1$.

(3) $C_2 = \{\zeta \mid \zeta = z_1 + i_2 z_2, z_1, z_2 \in C_1\}$.
Addition: $\zeta_1 + \zeta_2 = (z_1 + i_2 z_2) + (w_1 + i_2 w_2) = (z_1 + w_1) + i_2(z_2 + w_2)$.
Multiplication: $\zeta_1 \zeta_2 = (z_1 + i_1 z_2)(w_1 + i_1 w_2) = (z_1 w_1 - z_2 w_2) + i_2(z_1 w_2 + z_2 w_1)$, $i_2^2 = -1$.

(4) $C_3 = \{\xi \mid \xi = \zeta_1 + i_3 \zeta_2, \zeta_1, \zeta_2 \in C_2\}$.
Addition: $\xi_1 + \xi_2 = (\zeta_1 + i_3 \zeta_2)(\omega_1 + i_3 \omega_2) = (\zeta_1 + \omega_1) + i_3(\zeta_2 + \omega_2)$
Multiplication: $\xi_1 \xi_2 = (\zeta_1 + i_3 \zeta_2)(\omega_1 + i_3 \omega_2) = (\zeta_1 \omega_1 - \zeta_2 \omega_2) + i_3(\zeta_1 \omega_2 + \zeta_2 \omega_1)$, $i_3^2 = -1$.

(5) $C_n := \{\gamma \mid \gamma = \alpha_1 + i_n \alpha_2, \alpha_1, \alpha_2 \in C_{n-1}\} =$
Addition: $\gamma + \delta = (\alpha_1 + i_n \alpha_2) + (\beta_1 + i_n \beta_2) = (\alpha_1 + \beta_1) + i_n(\alpha_2 + \beta_2)$
Multiplication: $\gamma\delta = (\alpha_1 + i_n \alpha_2)(\beta_1 + i_n \beta_2) = (\alpha_1 \beta_1 - \alpha_2 \beta_2) + i_n(\alpha_1 \beta_2 + \alpha_2 \beta_1)$, $i_n^2 = -1$.

2.1. Cauchy-Riemann Matrix representation of Bicomplex numbers

Each of the two linear algebra representations x and z of the bicomplex algebra correspond to a matrix algebra which is isomorphic to C_2. These matrices are called Cauchy-Riemann matrices. In this section the steps for finding Cauchy-Riemann matrices will be shown.

Definition 2.7. Two algebras A and B are said to be isomorphic if and only if there exists a one-to-one mapping $\varphi : A \to B$, $a \mapsto \varphi(a)$, of A onto B such that

(1) $\varphi(a_1 + a_2) = \varphi(a_1) + \varphi(a_2)$,

(2) $\varphi(a_1 a_2) = \varphi(a_1)\varphi(a_2)$.

Multiplication of all of the elements in C_2 by an element $z_1 + i_2 z_2$ performs a linear transformation on C_2; this property of C_2 results from the properties of multiplication described in Theorem 2.5. If

$$(z_1 + i_2 z_2)(w_1 + i_2 w_2) = s_1 + s_2 i_2,$$

then

$$z_1 w_1 - z_2 w_2 = s_1, z_2 w_1 + z_1 w_2 = s_2 \quad (T).$$

Thus multiplication by $z_1 + i_2 z_2$ corresponds to the linear transformation (T) with matrix

$$\begin{bmatrix} z_1 & -z_2 \\ z_2 & z_1 \end{bmatrix}.$$

Theorem 2.8. *Let $z = z_1 + i_2 z_2$ be a bicomplex variable, and M be a function defined on C_2 as $M(z) = [z]$ such that*

$$[z] = \begin{bmatrix} z_1 & -z_2 \\ z_2 & z_1 \end{bmatrix}.$$

Then the set of 2×2 complex Cauchy-Riemann matrices $[z]$ with the operations of matrix addition and multiplication is isomorphic to the bicomplex algebra C_2.

Proof. Theorem 28.2, [8]. □

Theorem 2.9. *Let $x = x_1 + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4$ be an element in C_2, and let N be a function defined on C_2 as $N(x) = [x]$ such that:*

$$[x] = \begin{bmatrix} x_1 & -x_2 & -x_3 & x_4 \\ x_2 & x_1 & -x_4 & -x_3 \\ x_3 & -x_4 & x_1 & -x_2 \\ x_4 & x_3 & x_2 & x_1 \end{bmatrix}.$$

Then the set of 4×4 complex Cauchy-Riemann matrices $[x]$ with the operations of matrix addition and multiplication is isomorphic to the bicomplex algebra C_2.

Proof. Theorem 28.4, [8]. □

Example 2.10. Below are shown the Cauchy-Riemann matrices for i_1 and i_2:

$$i_1 \equiv \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad i_2 \equiv \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}. \tag{2.1}$$

Step 1. Assume $z_1 = x_1 + i_1x_2$, $z_2 = x_3 + i_1x_4$, $w_1 = x_5 + i_1x_6$ and $w_2 = x_7 + i_1x_8$ are complex variables. The Cauchy-Riemann matrices correspond to these complex variables are:

$$z_1 \leftrightarrow A = \begin{bmatrix} x_1 & -x_2 \\ x_2 & x_1 \end{bmatrix}, \quad z_2 \leftrightarrow B = \begin{bmatrix} x_3 & -x_4 \\ x_4 & x_3 \end{bmatrix}, \tag{2.2}$$

$$w_1 \leftrightarrow C = \begin{bmatrix} x_5 & -x_6 \\ x_6 & x_5 \end{bmatrix}, \quad w_2 \leftrightarrow D = \begin{bmatrix} x_7 & -x_8 \\ x_8 & x_7 \end{bmatrix}. \tag{2.3}$$

Step 2. We know that there are two different representations for bicomplex variables z and w as below:

$$z = z_1 + i_2z_2, \ w = w_1 + i_2w_2; \quad z_1, z_2, w_1, w_2 \in C_1,$$

$$z = (x_1 + i_1x_2) + i_2(x_3 + i_1x_4), \ w = (x_5 + i_1x_6) + i_2(x_7 + i_1x_8); \ x_1, ..., x_8 \in C_0.$$

Consequently, there are two different corresponding Cauchy-Riemann matrices as bellow:

$$z \leftrightarrow \begin{bmatrix} z_1 & -z_2 \\ z_2 & z_1 \end{bmatrix}, \quad w \leftrightarrow \begin{bmatrix} w_1 & -w_2 \\ w_2 & w_1 \end{bmatrix}.$$

Matrix representation of bicomplex variables $z = (x_1 + i_1x_2) + i_2(x_3 + i_1x_4)$ and $w = (x_5 + i_1x_6) + i_2(x_7 + i_1x_8)$ is accomplished by organizing the matrices A, B, C and D into the above corresponding Cauchy-Riemann matrices:

$$z \leftrightarrow E = \begin{bmatrix} A & -B \\ B & A \end{bmatrix}_{2^2 \times 2^2} = \begin{bmatrix} x_1 & -x_2 & -x_3 & x_4 \\ x_2 & x_1 & -x_4 & -x_3 \\ x_3 & -x_4 & x_1 & -x_2 \\ x_4 & x_3 & x_2 & x_1 , \end{bmatrix},$$

and similarly for w,

$$w \leftrightarrow F = \begin{bmatrix} C & -D \\ D & C \end{bmatrix}_{2^2\times 2^2} = \begin{bmatrix} x_5 & -x_6 & -x_7 & x_8 \\ x_6 & x_5 & -x_8 & -x_7 \\ x_7 & -x_8 & x_5 & -x_6 \\ x_8 & x_7 & x_6 & x_5 \end{bmatrix}.$$

Step 3. Following the same procedure, the matrix representation of a tricomplex variable $\zeta = x_1 + i_1x_2 + i_2x_3 + i_1i_2x_4 + i_3x_5 + i_1i_3x_6 + i_2i_3x_7 + i_1i_2i_3x_8$ is given as:

$$\zeta \leftrightarrow G = \begin{bmatrix} E & -F \\ F & E \end{bmatrix}_{2^3\times 2^3} = \begin{bmatrix} A & -B & -C & D \\ B & A & -D & -C \\ C & -D & A & -B \\ D & C & B & A \end{bmatrix}_{2^3\times 2^3}$$

$$= \begin{bmatrix} x_1 & -x_2 & -x_3 & x_4 & -x_5 & x_6 & x_7 & -x_8 \\ x_2 & x_1 & -x_4 & -x_3 & -x_6 & -x_5 & x_8 & x_7 \\ x_3 & -x_4 & x_1 & -x_2 & -x_7 & x_8 & -x_5 & x_6 \\ x_4 & x_3 & x_2 & x_1 & -x_8 & -x_7 & -x_6 & -x_5 \\ x_5 & -x_6 & -x_7 & x_8 & x_1 & -x_2 & -x_3 & x_4 \\ x_6 & x_5 & -x_8 & -x_7 & x_2 & x_1 & -x_4 & -x_3 \\ x_7 & -x_8 & x_5 & -x_6 & x_3 & -x_4 & x_1 & -x_2 \\ x_8 & x_7 & x_6 & x_5 & x_4 & x_3 & x_2 & x_1 \end{bmatrix}.$$

Using this procedure one can determine the $2^n \times 2^n$ Cauchy-Riemann Matrix representation of a multicomplex variable. These steps can be used to code an algorithm for finding Cauchy-Riemann Matrix representation of a multicomplex variable.

3. Multicomplex Taylor Series Expansion method

In this section, the Multicomplex Taylor Series Expansion(MCTSE) method for calculating numerically accurate the high-order derivatives of a given problem is presented. The MCTSE applies a Taylor series expansion to a multicomplex variable evaluated on the real axis at x and perturbed by $h(i_1 + \ldots + i_n)$ to drive the n^{th} derivative. Based on the MCTSE for computing the second order derivative of a function with respect to variable x, apply a Taylor series expansion to a bicomplex variable evaluated on the real axis at x and perturbed by

$h(i_1+i_2)$ as follows:

$f[x+h(i_1+i_2)]$
$= f(x)+h(i_1+i_2)f'(x)+\frac{h^2}{2}(i_1+i_2)^2 f''(x)+\frac{h^3}{3!}(i_1+i_2)^3 f'''(x)+O(h^4)$
$= f(x)+h(i_1+i_2)f'(x)+\frac{h^2}{2}(i_1^2+2i_1i_2+i_2^2)f''(x)+\frac{h^3}{3!}(i_1^3+3i_1^2i_2$
$+3i_1i_2^2+i_2^3)f'''(x)+O(h^4) = f(x)+h(i_1+i_2)f'(x)+h^2(i_1i_2)f''(x)-h^2f''(x)+$
$\frac{4h^3}{3!}(-i_1-i_2)f'''(x)+O(h^4)$,
where $i_1^2=i_2^2=-1$.

Consequently, taking the coefficient of i_1i_2 from both sides, $Im_{12}(f[x+h(i_1+i_2)]) = h^2f''(x)+O(h^4)$. Therefore, the formula for computing the second order derivative can be achieved as follows:

$$f''(x)=\frac{Im_{12}(f[x+h(i_1+i_2)])}{h^2}+O(h^2),$$

where $O(h^2)$ represent the order of the error. The perturbation h can be set to the smallest value that the computer can take, e.g. 10^{-50}. The main advantage of the MCTSE method is that it contains no subtractive cancelation error, and therefore the truncation error can be made arbitrarily small.

The MCTSE applies a Taylor series expansion of a holomorphic function f to a multicomplex variable evaluated on the real axis at x and perturbed by $h(i_1+\ldots+i_n)$ to drive the n^{th} derivative of f as follows:

$$f(x+hi_1+\ldots+hi_n)\approx f(x)+(i_1+\ldots+i_n)hf'(x)+(i_1+\ldots+i_n)^2h^2\frac{f''(x)}{2}+\ldots$$

$$+(i_1+\ldots+i_n)^nh^n\frac{f^{(n)}(x)}{n!}+(i_1+\ldots+i_n)^{n+1}h^{n+1}\frac{f^{(n+1)}(x)}{(n+1)!}.$$

Based on the multinomial theorem, for $k_1,\ldots,k_n; k_1+\ldots+k_n=k$:

$$(i_1+\ldots+i_n)^k=\sum_{k_1,\ldots,k_n}\frac{n!}{k_1!\ldots k_n!}i_1^{k_1}\ldots i_n^{k_n}.$$

Consequently, taking the coefficient of $i_1\ldots i_n$ from both sides, we have $Im_{1\ldots n}(f[x+hi_1+\ldots+hi_n]) = f^{(n)}(x)h^n+O(h^2)$, where $Im_{1\ldots n}$ is the coefficient of $i_1\ldots i_n$ in Taylor series expansion of $f(x+hi_1+\ldots+hi_n)$. Therefore, the formula for computing the high-order derivative can be achieved as follows:

$$f^{(n)}(x)=\frac{Im_{1\ldots n}(f[x+hi_1+\ldots+hi_n)])}{h^n}+O(h^2).$$

In general, extending this result to obtain the n^{th}-order partial derivatives of any holomorphic functions of m variables is as follows:

$$\frac{\partial^n f(x_1,\dots x_m)}{\partial x_1^{b_1}\dots\partial x_m^{b_m}} \approx \frac{Im_{1\dots n}[f(x_1+h\sum_{j=1}^{b_1} i_j,\dots,x_m+h\sum_{j=1}^{b_m} i_j)]}{h^n},$$

where $\sum_{i=1}^{m} b_i = n$.

Let $f(x,y)$ be a two variable function. Computation of the first two and partial derivatives with respect to x and y is possible through perturbing the function in both directions of i_1 and i_2 in the bicomplex space. The perturbation h can be extremely small (e.g. 10^{-30}) because there is no subtraction errors, unlike typical numerical differencescheme. The formulas for calculating the first two derivatives of a two variable function based on MCTSE method is shown below:

$$\frac{\partial f(x,y)}{\partial x} \approx \frac{Im_1(f(x+hi_1+hi_2,y))}{h} = \frac{Im_2(f(x+hi_1+hi_2,y))}{h}, \tag{3.1}$$

$$\frac{\partial f(x,y)}{\partial y} \approx \frac{Im_1(f(x,y+hi_1+hi_2))}{h} = \frac{Im_2(f(x,y+hi_1+hi_2))}{h}, \tag{3.2}$$

$$\frac{\partial^2 f(x,y)}{\partial x^2} \approx \frac{Im_{12}(f[x+hi_1+hi_2,y])}{h^2}, \tag{3.3}$$

$$\frac{\partial^2 f(x,y)}{\partial y^2} \approx \frac{Im_{12}(f[x,y+hi_1+hi_2])}{h^2}, \tag{3.4}$$

$$\frac{\partial^2 f(x,y)}{\partial x \partial y} \approx \frac{Im_{12}(f[x+hi_1,y+hi_2])}{h^2}. \tag{3.5}$$

Therefore, for finding the first two derivatives with respect to one variable, x and y are replaced with their perturbed values in two directions, $Z_x^{12} = x+hi_1+hi_2$ and $Z_y^{12} = y+hi_1+hi_2$, respectively. Also to find the partial derivative of a two variable function, x and y are replaced with their perturbed values in two directions, $Z_x^1 = x+i_1h$ and $Z_y^2 = y+i_2h$, respectively. The matrix representations of the bicomplex numbers Z_x^{12}, Z_y^{12},Z_x^1, and Z_y^2 are given below:

$$Z_x^{12} = x+hi_1+hi_2 \leftrightarrow M_x^{12} = \begin{bmatrix} x & -h & -h & 0 \\ h & x & 0 & -h \\ h & 0 & x & -h \\ 0 & h & h & x \end{bmatrix},$$

$$Z_y^{12} = y + hi_1 + hi_2 \leftrightarrow M_y^{12} = \begin{bmatrix} y & -h & -h & 0 \\ h & y & 0 & -h \\ h & 0 & y & -h \\ 0 & h & h & y \end{bmatrix},$$

$$Z_x^1 = x + hi_1 \leftrightarrow M_x^1 = \begin{bmatrix} x & -h & 0 & 0 \\ h & x & 0 & 0 \\ 0 & 0 & x & -h \\ 0 & 0 & h & x \end{bmatrix},$$

$$Z_y^2 = y + hi_2 \leftrightarrow M_y^2 = \begin{bmatrix} y & 0 & -h & 0 \\ 0 & y & 0 & -h \\ h & 0 & y & 0 \\ 0 & h & 0 & y \end{bmatrix}.$$

For any holomorphic function f of two variables let:

$$F_1 = [f(M_x^{12}, y)], \quad F_2 = [f(x, M_y^{12})], \quad F_3 = [f(M_x^1, M_y^2)].$$

Based on formulas (1) – (5), the first and second derivatives of f are computed by the following formulas:

$$\frac{\partial f(x,y)}{\partial x} \approx \frac{[F_1]_{21}}{h} = \frac{[F_1]_{31}}{h} = \frac{[F_3]_{21}}{h},$$

$$\frac{\partial f(x,y)}{\partial y} \approx \frac{[F_2]_{21}}{h} = \frac{[F_2]_{31}}{h} = \frac{[F_3]_{31}}{h},$$

$$\frac{\partial^2 f(x,y)}{\partial x^2} \approx \frac{[F_1]_{41}}{h^2}, \quad \frac{\partial^2 f(x,y)}{\partial y^2} \approx \frac{[F_2]_{41}}{h^2}, \quad \frac{\partial^2 f(x,y)}{\partial x \partial y} \approx \frac{[F_3]_{41}}{h^2},$$

where $[F_l]_{ij}$, denotes the i^{th} row, j^{th} column element of the matrix F_l, for all $1 \le l \le 3$, $1 \le i, j \le 4$. So if a_{kl}, $1 \le k, l \le 4$ be the element in the k^{th} row, l^{th} column of F_i, then:

$$a_{11} = The\ function\ value\,,$$

$$\frac{a_{21}}{h} = \frac{a_{31}}{h} = The\ first\ partials\,,$$

$$\frac{a_{41}}{h^2} = The\ second\ partials\,.$$

Let $f(x, y)$ be a two variable function. Computation of the first three derivatives with respect to x and y is possible through perturbing the function in all three directions of i_1, i_2 and i_3 in the tricomplex space. Therefore, for finding the first three derivatives with respect to one variable, x and y are replaced with their perturbed values in three directions, $\xi_x^{123} = x + i_1h + i_2h + i_3h$ and $\xi_y^{123} = y + i_1h + i_2h + i_3h$, respectively. Also to find the partial derivatives, we perturb the function in the combination of all three directions of i_1, i_2 and i_3, such as $x + hi_1 + hi_3$ and $y + hi_2$. So two pairs of tricomplex variables, ξ_x^{13}, ξ_y^2, and ξ_x^1, ξ_y^{23} can be considered, respectively as follows:

$$\begin{cases} \xi_x^{13} = x + i_1h + i_3h \\ \xi_y^2 = y + i_2h \end{cases},$$

$$\begin{cases} \xi_x^1 = x + i_1h \\ \xi_y^{23} = y + i_2h + i_3h \end{cases}.$$

It should be noted that all derivatives up to the order three can be computed using tricomplex variables. The matrices representation of the tricomplex numbers ξ_x^{123}, ξ_y^{123}, ξ_x^{13}, ξ_y^2, ξ_x^1, and ξ_y^{23} are given below:

$$\xi_x^{123} = x+i_1h+i_2h+i_3h \leftrightarrow N_x^{123} = \begin{bmatrix} x & -h & -h & 0 & -h & 0 & 0 & 0 \\ h & x & 0 & -h & 0 & -h & 0 & 0 \\ h & 0 & x & -h & 0 & 0 & -h & 0 \\ 0 & h & h & x & 0 & 0 & 0 & -h \\ h & 0 & 0 & 0 & x & -h & -h & 0 \\ 0 & h & 0 & 0 & h & x & 0 & -h \\ 0 & 0 & h & 0 & h & 0 & x & -h \\ 0 & 0 & 0 & h & 0 & h & h & x \end{bmatrix},$$

$$\xi_y^{123} = y+i_1h+i_2h+i_3h \leftrightarrow N_y^{123} = \begin{bmatrix} y & -h & -h & 0 & -h & 0 & 0 & 0 \\ h & y & 0 & -h & 0 & -h & 0 & 0 \\ h & 0 & y & -h & 0 & 0 & -h & 0 \\ 0 & h & h & y & 0 & 0 & 0 & -h \\ h & 0 & 0 & 0 & y & -h & -h & 0 \\ 0 & h & 0 & 0 & h & y & 0 & -h \\ 0 & 0 & h & 0 & h & 0 & y & -h \\ 0 & 0 & 0 & h & 0 & h & h & y \end{bmatrix},$$

$$\xi_x^{13} = x + i_1 h + i_3 h \leftrightarrow N_x^{13} = \begin{bmatrix} x & -h & 0 & 0 & -h & 0 & 0 & 0 \\ h & x & 0 & 0 & 0 & -h & 0 & 0 \\ 0 & 0 & x & -h & 0 & 0 & -h & 0 \\ 0 & 0 & h & x & 0 & 0 & 0 & -h \\ h & 0 & 0 & 0 & x & -h & 0 & 0 \\ 0 & h & 0 & 0 & h & x & 0 & 0 \\ 0 & 0 & h & 0 & 0 & 0 & x & -h \\ 0 & 0 & 0 & h & 0 & 0 & h & x \end{bmatrix},$$

$$\xi_y^{2} = y + i_2 h \leftrightarrow N_y^{2} = \begin{bmatrix} y & 0 & -h & 0 & 0 & 0 & 0 & 0 \\ 0 & y & 0 & -h & 0 & 0 & 0 & 0 \\ h & 0 & y & 0 & 0 & 0 & 0 & 0 \\ 0 & h & 0 & y & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & y & 0 & -h & 0 \\ 0 & 0 & 0 & 0 & 0 & y & 0 & -h \\ 0 & 0 & 0 & 0 & h & 0 & y & 0 \\ 0 & 0 & 0 & 0 & 0 & h & 0 & y \end{bmatrix},$$

$$\xi_x^{1} = x + i_1 h \leftrightarrow N_x^{1} = \begin{bmatrix} x & -h & 0 & 0 & 0 & 0 & 0 & 0 \\ h & x & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x & -h & 0 & 0 & 0 & 0 \\ 0 & 0 & h & x & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x & -h & 0 & 0 \\ 0 & 0 & 0 & 0 & h & x & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & x & -h \\ 0 & 0 & 0 & 0 & 0 & 0 & h & x \end{bmatrix},$$

$$\xi_y^{23} = y + i_2 h + i_3 h \leftrightarrow N_y^{23} = \begin{bmatrix} y & 0 & -h & 0 & -h & 0 & 0 & 0 \\ 0 & y & 0 & -h & 0 & -h & 0 & 0 \\ h & 0 & y & 0 & 0 & 0 & -h & 0 \\ 0 & h & 0 & y & 0 & 0 & 0 & -h \\ h & 0 & 0 & 0 & y & 0 & -h & 0 \\ 0 & h & 0 & 0 & 0 & y & 0 & -h \\ 0 & 0 & h & 0 & h & 0 & y & 0 \\ 0 & 0 & 0 & h & 0 & h & 0 & y \end{bmatrix}.$$

For any holomorphic function f of two variables let:

$$G_1 = [f(N_x^{123}, y)],\ \ G_2 = [f(x, N_y^{123})],\ \ G_3 = [f(N_x^{13}, N_y^{2})],\ \ G_4 = [f(N_x^{1}, N_y^{23})].$$

The following formula shows the first third derivatives of the function f with respect to x and y, where $[G_l]_{ij}$, $1 \leq l \leq 3$, $1 \leq i,j \leq 4$ denotes the i^{th} row, j^{th} column element of the matrix G_l:

$$\frac{\partial f(x,y)}{\partial x} \approx \frac{[G_1]_{21}}{h} = \frac{[G_1]_{31}}{h} = \frac{[G_1]_{51}}{h},$$

$$\frac{\partial^2 f(x,y)}{\partial x^2} \approx \frac{[G_1]_{41}}{h^2} = \frac{[G_1]_{61}}{h^2} = \frac{[G_1]_{71}}{h^2},$$

$$\frac{\partial^3 f(x,y)}{\partial x^3} \approx \frac{[G_1]_{81}}{h^3}, \quad \frac{\partial^3 f(x,y)}{\partial y^3} \approx \frac{[G_2]_{81}}{h^3},$$

$$\frac{\partial^3 f(x,y)}{\partial x^2 \partial y} \approx \frac{[G_3]_{81}}{h^3}, \quad \frac{\partial^3 f(x,y)}{\partial x \partial y^2} \approx \frac{[G_4]_{81}}{h^3}.$$

In general, extending this result to obtain the n^{th}-order partial derivatives of any holomorphic function of m variables is as follows:

$$\frac{\partial^n f(x_1, \dots x_m)}{\partial x_1^{b_1} \dots \partial x_m^{b_m}} \approx \frac{[g(N_{x_1}^{k_1}, \dots g(N_{x_m}^{k_m})]_{n1}}{h^n},$$

where $k_i \subseteq \{0, 1, \dots n\}$, $|k_i| = b_i$, $1 \leq i \leq m$, and the symbol $|.|$ shows the number of nonzero elements of the set k_i. Also $\sum_{i=1}^{m} |k_i| = \sum_{i=1}^{m} b_i = n$ and $N_{x_1}^{0} = x_1$, which means x_1 has not been perturbed in any direction. For instance, $N_{x_1}^{02} = N_{x_1}^{2}$ means x_1 has been perturbed in only i_2 direction. So $N_{x_1}^{02}$ is a $2^n \times 2^n$ matrix representation of $x_1 + i_2 h$. In this case, the number of derivatives is $\binom{n+m-1}{m-1}$. Using tricomplex variables for computing the third derivative of a three variables function with respect to x, y and z, the first column of the matrix $B = f[(M_x^1, M_y^2, M_z^3)]_{2^3 \times 2^3}$ should be considered. So if b_{mn}, $1 \leq m,n \leq 8$ be the element in the m^{th} row, n^{th} column of B, then:

$$b_{11} = The\ function\ value\,,$$

$$\frac{b_{21}}{h} = \frac{b_{31}}{h} = \frac{b_{51}}{h} = The\ first\ partials\,,$$

$$\frac{b_{41}}{h^2} = \frac{b_{61}}{h^2} = \frac{b_{71}}{h^2} = The\ second\ partials\,,$$

$$\frac{b_{81}}{h^3} = The\ third\ partials\,.$$

Using quadcomplex variables for computing the fourth derivative of a four variables function with respect to x, y, z and w, the first column of the matrix

$C = [f(M_x^1, M_y^2, M_z^3, M_w^4)]_{2^4\times 2^4}$ should be considered. So if c_{ep}, $1 \leq e, p \leq 16$ is the element in the e^{th} row, p^{th} column of C, then:

$$c_{11} = The\ function\ value$$

$$\frac{c_{21}}{h} = \frac{c_{31}}{h} = \frac{c_{51}}{h} = \frac{c_{91}}{h} = The\ first\ partials\,,$$

$$\frac{c_{41}}{h^2} = \frac{c_{61}}{h^2} = \frac{c_{71}}{h^2} = \frac{c_{101}}{h^2} = \frac{c_{111}}{h^2} = \frac{c_{131}}{h^2} = The\ second\ partials\,,$$

$$\frac{c_{81}}{h^3} = \frac{c_{121}}{h^3} = \frac{c_{141}}{h^3} = \frac{c_{151}}{h^3} = The\ third\ partials\,,$$

$$\frac{c_{161}}{h^4} = The\ fourth\ partials\,.$$

As proven through the above process, the number of the k^{th} derivatives for $k = 0, 1, 2, \ldots$ is the coefficient of the binomial series:

$$\begin{array}{ccccccccccccc}
 & & & & & & 1f & & C_0 & & & & \\
 & & & & & 1f & & 1f' & & C_1 & & & \\
 & & & & 1f & & 2f' & & 1f'' & & C_2 & & \\
 & & & 1f & & 3f' & & 3f'' & & 1f''' & & C_3 & \\
 & & 1f & & 4f' & & 6f'' & & 4f''' & & 1f^{(4)} & & C_4 \\
1f & & 5f' & & 10f'' & & 10f''' & & 5f^{(4)} & & 1f^{(5)} & & C_5 \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots
\end{array}$$

The following example shows the implementation of MCTSE method one step at a time.

Example 3.1. Using the MCTSE method the first two derivatives of $f(x) = x^3$ will be computed:

$$f(x + hi_1 + hi_2) = x^3 + h(i_1 + i_2)(3x^2) + \frac{h^2}{2!}(i_1 + i_2)^2 + \frac{h^3}{3!}(i_1 + i_2)^3 6 + o(h^4),$$

$$f'(x) \approx \frac{Im_1[f(x + hi_1)]}{h} = \frac{3x^2 h}{h},$$

$$f''(x) \approx \frac{Im_{12}[f(x + hi_1 + hi_2)]}{h^2} = \frac{6xh^2}{h^2}.$$

Example 3.2 shows the implementation of the MCTSE for the function $f(x) = x^3$ using the Cauchy-Riemann matrix representation, which facilitates the process.

Example 3.2. Based on the MCTSE, using Cauchy-Riemann matrix representation (1) and evaluating $f(M_x^{12})$, we have:

$$f(M_x^{12}) = \begin{pmatrix} x^3 - 6h^2x & 4h^3 - 3hx^2 & 4h^3 - 3hx^2 & 6h^2x \\ 3hx^2 - 4h^3 & x^3 - 6h^2x & -6h^2x & 4h^3 - 3hx^2 \\ 3hx^2 - 4h^3 & -6h^2x & x^3 - 6h^2x & 4h^3 - 3hx^2 \\ 6h^2x & 3hx^2 - 4h^3 & 3hx^2 - 4h^3 & x^3 - 6h^2x \end{pmatrix}.$$

It should be noticed that the element of the first row, first column of $f(M_x^{12})$ is always function value. The elements of the second and third rows, first column divided by h are the first derivatives. Finally, the element of the fourth row, first column divided by h^2 is the second derivative of the function. So we have:

$$f'(x) = \frac{[f(M_x^{12})]_{21}}{h} = \frac{[f(M_x^{12})]_{31}}{h} = \frac{3hx^2}{h} = 3x^2,$$

$$f''(x) = \frac{[f(M_x^{12})]_{41}}{h^2} = \frac{6h^2x}{h^2} = 6x.$$

3.1. Where is MCTSE method not applicable?

Definition 3.3. A function f is real analytic on an open set D in the real line if for any x_0 in D one can write:

$$f(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + \dots,$$

in which the coefficients $a_0, a_1, \dots$ are real numbers and the series is convergent to $f(x)$ for x in a neighborhood of x_0.

Equivalently, an analytic function f is infinitely differentiable and the Taylor series at the point x_0 in its domain, converges to $f(x)$ for all x in a neighborhood of x_0.

Definition 3.4. A function that has derivatives of all orders is called smooth.

Warner [11], also defined the analytic function as follows:

Definition 3.5. The function f is said to be analytic if it is smooth and if it equals its Taylor series expansion about any point in its domain.

For instance, the exponential and trigonometric functions are typically analytic in their domain. Some properties of analytic functions are shown here:

(1) The sums, products, and compositions of analytic functions are analytic.

(2) The reciprocal of an analytic function that is non-zero is analytic, as is the inverse of an invertible analytic function whose derivative is non-zero.

(3) Any analytic function is smooth, that is, infinitely differentiable, but the converse is not true.

A function f defined on some subset of the real line is said to be real analytic at a point x if there is a neighborhood D of x on which f is real analytic. The definition of a complex analytic function is obtained by replacing, in the above definition, real with complex and real line with complex plane.

Definition 3.6. A complex-valued function of a complex variable z is said to be holomorphic at a point a if it is differentiable at every point within some open disk centered at a, and is said to be analytic at a if in some open disk centered at a it can be expanded as a convergent power series

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$$

this implies that the radius of convergence is positive.

Theorem 3.7. *Every holomorphic function is analytic.*

Since non-analytic functions do not have Taylor series expansion, the MCTSE method is not applicable for them. Example 3.8, introduces some non-analytic functions.

Example 3.8. $|x|$ and $\exp(-1/x^2)$ are not analytic at the zero, so the MCTSE method is not applicable for them. Also, the Fourier series $f(x) :=$

$\sum_{k\in A}\exp(-\sqrt{k})\cos(kx)$, which is an infinitely differentiable function is not analytic at any point.

4. Use of Multicomplex Mathematics for the Calculation of High-Order Derivatives of Complex-Valued Functions

In this section we will show that Complex Taylor Series Expansion (MCTSE) method can be applied for computing high-order derivatives of complex-valued functions.

Krantz [4], formulated the notion of holomorphic functions in terms of the real and imaginary parts of f: Let $U \subseteq C_1$ be a region and $f : U \to C_1$ a differentiable function. Write

$$f(z) = u_1(x, y) + i_1 v_1(x, y)$$

with u_1 and v_1 real-valued functions. Of course $z = x + i_1 y$ as usual. If u_1 and v_1 satisfy the Cauchy-Riemann equations:

$$\frac{\partial u_1}{\partial x} = \frac{\partial v_1}{\partial y} \quad \textit{and} \quad \frac{\partial u_1}{\partial y} = -\frac{\partial v_1}{\partial x},$$

at every point of U, then the function f is said to be holomorphic.

Definition 4.1. Assume $f = u_1 + i_1 v_1 : U \to C_1$ is a differentiable function. Then we have:

$$\frac{\partial}{\partial z} f \equiv \frac{1}{2}(\frac{\partial}{\partial x} - i_1 \frac{\partial}{\partial y}) f = \frac{1}{2}(\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y}) + \frac{i_1}{2}(\frac{\partial v_1}{\partial x} - \frac{\partial u_1}{\partial y}),$$

$$\frac{\partial}{\partial \bar{z}} f \equiv \frac{1}{2}(\frac{\partial}{\partial x} + i_1 \frac{\partial}{\partial y}) f = \frac{1}{2}(\frac{\partial u_1}{\partial x} - \frac{\partial v_1}{\partial y}) + \frac{i_1}{2}(\frac{\partial v_1}{\partial x} + \frac{\partial u_1}{\partial y}).$$

Now, we will generalize Definition 4.1, for bicomplex variables.

Definition 4.2. Let X be a domain in C_2, and $f : X \to C_2$ be a holomorphic function in X. If $\zeta = x_1 + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4$ and $x = (x_1, \ldots, x_4)$ then f has the following isomorphic representations:

$$f(\zeta) = g_1(x) + i_1 g_2(x) + i_2 g_3(x) + i_1 i_2 g_4(x), \quad g : X \to C_0^4,$$

$$f(\zeta) = u(z_1, z_2) + i_2 v(z_1, z_2), \quad u : X \to C_1,\ v : X \to C_1,$$

in which

$$u(z_1, z_2) = u_1(x, y) + i_1 u_2(x, y),$$

and

$$v(z_1, z_2) = v_1(x, y) + i_1 v_2(x, y).$$

So we have:

$$\frac{\partial}{\partial \zeta} f \equiv \frac{1}{2}(\frac{\partial}{\partial z_1} - i_2 \frac{\partial}{\partial z_2}) f = \frac{1}{2}(\frac{\partial u}{\partial z_1} + \frac{\partial v}{\partial z_2}) + \frac{i_2}{2}(\frac{\partial v}{\partial z_1} - \frac{\partial u}{\partial z_2}),$$

$$\frac{\partial}{\partial \bar{\zeta}} f \equiv \frac{1}{2}(\frac{\partial}{\partial z_1} + i_2 \frac{\partial}{\partial z_2}) f = \frac{1}{2}(\frac{\partial u}{\partial z_1} - \frac{\partial v}{\partial z_2}) + \frac{i_2}{2}(\frac{\partial v}{\partial z_1} + \frac{\partial u}{\partial z_2}).$$

We know that if f is any complex-valued function, then we may write $f = u_1 + iv_1$, where u_1 and v_1 are real-valued functions. We can apply Complex Taylor Series Expansion (MCTSE) method for the calculation of high-order derivatives of complex-valued functions. For this purpose, we need to apply MCTSE method for finding the high-order derivatives of real-valued functions u_1 and v_1. Therefore, we can calculate the high-order derivatives of complex-valued functions by generalizing Definition 4.1 to the k^{th} derivative. In other words, we apply MCTSE method to real-valued functions u_1 and v_1 through the following formula:

$$\frac{\partial^k}{\partial z^k} f \equiv \frac{1}{2}(\frac{\partial^k}{\partial x^k} - i \frac{\partial^k}{\partial y^k}) f = \frac{1}{2}(\frac{\partial^k u_1}{\partial x^k} + \frac{\partial^k v_1}{\partial y^k}) + \frac{i}{2}(\frac{\partial^k v_1}{\partial x^k} - \frac{\partial^k u_1}{\partial y^k}),$$

$$\frac{\partial^k}{\partial \bar{z}^k} f \equiv \frac{1}{2}(\frac{\partial^k}{\partial x^k} + i \frac{\partial^k}{\partial y^k}) f = \frac{1}{2}(\frac{\partial^k u_1}{\partial x^k} - \frac{\partial^k v_1}{\partial y^k}) + \frac{i}{2}(\frac{\partial^k v_1}{\partial x^k} + \frac{\partial^k u_1}{\partial y^k}).$$

Krantz [4], states the following theorem for calculating the high-order derivatives of complex-valued functions without using the analytical methods.

Theorem 4.3. *Let $U \subseteq C_1$ be an open set and let f be holomorphic on U. Then $f \in C^\infty(U)$, (infinite times continuously differentiable). Moreover, if $\bar{D}(P, r) \subseteq U$ and $z \in D(P, r)$, then*

$$(\frac{d}{dz})^k f(z) = \frac{k!}{2\pi i} \oint_{|\zeta - P| = r} \frac{f(\zeta)}{(\zeta - z)^{k+1}} d\zeta, \qquad k = 0, 1, 2, \ldots$$

Proof. Page 94, [4]. □

MCTSE method seems to have a simpler structure compared to Krantz's method. However, further investigation is required to identify the superior method in terms of the accuracy.

Example 4.4. Let $f(z) = z^2 - z = (x + i_1 y)^2 - (x + i_1 y) = (x^2 - y^2 - x) + i_1(2xy - y) \equiv u(x, y) + i_1 v(x, y)$.

The following formula can be used to calculate the first and second derivatives of complex-valued function f:

$$\frac{\partial^2}{\partial z^2} f \equiv \frac{1}{2}\left(\frac{\partial^2}{\partial x^2} - i\frac{\partial^2}{\partial y^2}\right) f = \frac{1}{2}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + \frac{i}{2}\left(\frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 u}{\partial y^2}\right),$$

and using our algorithm in Matlab, we can get the high-order derivative of functions easily:

$$\frac{\partial u(x,y)}{\partial x} \approx \frac{[u_{21}(M_x, y)]}{h} = 2x - 1,$$

$$\frac{\partial u(x,y)}{\partial y} \approx \frac{[u_{21}(x, M_y)]}{h} = -2y,$$

$$\frac{\partial^2 u(x,y)}{\partial x^2} \approx \frac{[u_{41}(M_x, y)]}{h^2} = 2,$$

$$\frac{\partial^2 u(x,y)}{\partial y^2} \approx \frac{[u_{41}(x, M_y)]}{h^2} = -2.$$

In the above, $[u_{21}(M_x, y)]$ and $[u_{21}(x, M_y)]$ denote the second row, first column element of matrices $[u(M_x, y)]$ and $[u(x, M_y)]$, and $[u_{41}(M_x, y)]$ and $[u_{41}(x, M_y)]$ denote the fourth row, first column element of matrices $[u(M_x, y)]$ and $[u(x, M_y)]$. Similarly, one may calculate the first and second derivatives of $v = 2xy - y$. The second derivative of $f(z) = z^2 - z \equiv u(x, y) + i_1 v(x, y)$ is as follows:

$$\frac{\partial^2}{\partial z^2} f \equiv \frac{1}{2}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + \frac{i}{2}\left(\frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 u}{\partial y^2}\right) = \frac{1}{2}(2 + 0) + \frac{i}{2}(0 + 2) = 1 + i.$$

5. Conclusions

MCTSE has vast applications in problems that require calculating higher-order derivatives including, non-linear optimization, control and sensitivity analysis. In this paper, we investigated steps of the MCTSE method for calculating higher-order derivatives and showed the detail calculation of Cauchy-Riemann matrix for finding derivatives up to the third order. We also gave the general formula for calculating the n^{th} derivative. In addition, we showed that the number of times each derivative is calculated in the MCTSE method follows binomial series. Furthermore, we provided examples that show the MCTSE is not applicable for non-analytic functions. Finally, we discussed the application of the method for complex-valued functions.

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